An Explanation for Bankruptcy's Risk-Return Paradox

Carlos A. Mello-e-Souza

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INSEAD, Boulevard de Constance, 77300 Fontainebleau, France. Telephone: (331) 64.69.44.90.
Fax: (331) 64.22.77.31. E-mail: carlos.mello.e.souza@insead.fr.

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* INSEAD, Boulevard de Constance, 77300 Fontainebleau Cedex, France. Telephone: (331) 64.69.44.90. Fax: (331) 64.22.77.31. E-mail: carlos.mello.e.souza@insead.fr. This research was made possible by Insead and CEDEP's generous offer of facilities, computers and databases during the Spring and Summer of 2001. For this I wish to express my sincerest gratitude to Deigan Morris, David Young and all my colleagues at Insead's Accounting and Control Area. Excellent systems support was provided by William Fisk and Vinay Shet of Insead's Area of Academic Computing. This paper benefitted from comments offered by Pat O'Brien, Robert Bowen, Massimo Massa and by workshop participants at York University, Universidade Nova de Lisboa and Insead. Responsibility for any remaining errors is mine.

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The admission of bankruptcy risk into a single-period CAPM economy leads to two surprising results. First, OLS beta is a biased measure of systematic risk. Second, the expected return conditional on the market is no longer a linear function of beta alone. Instead, it is a linear function of the ratio of conditional to unconditional expected dividends and of that ratio times beta. In this economy very distressed firms should underperform the market because of their high option values and strong contributions to portfolio diversification. This is confirmed by empirical tests that replicate Dichev's paradox.

In a recent study linking probability of bankruptcy with future returns, Dichev [1998] found that portfolios of financially distressed firms underperformed the market between 1981 and 1995. Given that a relatively large number of those firms should have been expected by investors to fail within a year after portfolio formation, intuition suggests that investors should have demanded higher than average returns to hold such portfolios.[1] More formally, if investors hold diversified portfolios of securities -- and therefore are not rewarded or penalized for carrying unsystematic risk -- then Dichev's result can be interpreted as implying either that the market is efficient but bankruptcy risk is systematic and lower than average, or that the market is inefficient (i.e., investors do not interpret correctly the impact of publicly available information related to the likelihood of bankruptcy) and accept a lower rate of return on distressed firms than they should. There are problems with both interpretations. The problem with the first interpretation is that ordinary least squares (OLS) estimates of systematic risk for very distressed firms in the NYSE-AMEX are higher than the overall average by 33.7 % (in the Nasdaq by 30.2 %) during the period from January 1981 until January 1999, which agrees with the notion that investors should demand higher than average returns to hold stock of distressed firms. The problem with the second interpretation is that, as Dichev demonstrates, sub-average returns to distressed firms persist far too long for a story based on market inefficiency to be plausible.

Motivated by Dichev's paradox, I investigate in this paper, from the perspective of a single-period CAPM economy, the risk and return implications of taking into account the possibility that firms can fail when shareholders have limited liability. The argument is based on an analysis of the covariance of firm-specific stock returns with the market and of return expectations conditional or not on the market. This enables me to reach conclusions which, although surprising within the traditional context where bankruptcy is not explicitly allowed, are in agreement with Dichev's results.

One implication of a valuation model that takes risk of failure into account is that systematic risk increases without limit as the probability of bankruptcy approaches one, but only if the linkage between the firm's operating cash flows and the market does not collapse towards zero at the same time. Another implication is that the OLS estimate of beta is in general biased and the bias depends on the probability of bankruptcy and on the correlation between operating cash flows and the market. Finally, although expected returns unconditional on the market are still a linear function of systematic risk alone (measured without bias), conditional expected returns are a linear function of the ratio of conditional to unconditional expected dividends and of that ratio times beta.

In this paper I present results of empirical tests performed to verify these implications with a sample of industrial firms incorporated in the United States, listed in the NYSE, AMEX or Nasdaq, covering the period 1973-1998. Regarding the relatively smaller and riskier Nasdaq firms in the sub-period 1981-1998 (when risk of bankruptcy was historically high) I find that the mean correlation with the market for firms in the most distressed of ten portfolios is only 5.6% (versus 38.4% for all portfolios), that the mean OLS beta is 1.42 (versus .58 after adjusting for risk of failure), and that expected year-ahead returns according to the standard CAPM are 2.3 percentage points *above* average (versus 3.1 points *below* average after adjusting for risk of failure.) Although the actual performance of distressed firms in this sub-sample is considerably worse than even the bankruptcy-adjusted model is capable of predicting (at 9.8 points below average), the revised CAPM predicts a performance that is also substantially below average and is 5.4 points lower than the traditional CAPM's figure.

The explanation to Dichev's paradox is therefore based on the dual empirical finding -- supported by a version of the CAPM that admits bankruptcy -- that (i) for the components of the most distressed firms' portfolio the linkage between cash flows and the market drops sufficiently to dominate the effect of the higher failure probabilities on systematic risk and (ii) return expectations given the market according to the bankruptcy-adjusted model are lower than they would be according to the traditional model due to the lower beta and to shareholders' option to abandon. In other words, it is entirely consistent with equilibrium prices in efficient capital markets that very distressed firms should underperform the market, due to their strong contributions to portfolio diversification combined with their significantly asymmetrical upside and downside risks.

Section I presents a summary of the literature leading to bankruptcy's risk-return paradox. Section II develops expressions for systematic risk and both conditional and unconditional expected returns in a single period model of the firm that allows for limited liability and for the possibility of bankruptcy. Section III compares the theoretical values of systematic risk and of expected returns with their traditional counterparts, indicates how apparently anomalous behavior of returns can be caused by overlooking bankruptcy, and proposes an estimator of beta that adjusts for the bias in OLS beta. Section IV describes the sample and the empirical tests employed. Section V presents the results and Section VI concludes.

I. Background

The concept of systematic risk has played a central role in finance, since Markowitz [1952] formalized the notion that one should hold a diversified portfolio under uncertainty. Sharpe [1964], Lintner [1965], and Mossin [1966] derived the CAPM by assuming that investors hold diversified portfolios, and predicted that only beta is priced and that beta and expected returns are linearly related. As research on the determinants and interplay of risk and return proceeded, a number of results began to challenge the central predictions of the CAPM. Prime examples are the January (Rozeff and Kinney [1976]), weekend (French [1980]), size (Banz [1981]), E/P (Basu [1977, 1983]), book-to-market ([Stattman [1980] and Rosenberg, Reid, and Lamstein [1985]), and leverage effects (Bhandari [1988]). Although anomalies such as the January effect have been detected in the United States since the early 1900's (Lakonishok and Smidt [1988]), the strength and direction of these effects has not been constant over time, as Brown, Kleidon, and Marsh [1983] showed regarding the size effect.

Reinganum [1982] tested Roll's [1980] conjecture that excessive small firm returns are due to improper estimation of systematic risk, but found that only a small portion of the size anomaly would go away. The E/P effect was found to be driven by size (Reinganum [1981]), but Basu [1983] countered with evidence that the size and P/E effects coexist. Almost all of the size effect happens in January, but the January effect is also more pronounced for the smallest firms (Keim [1983], and Reinganum [1983]). Bhardwaj and Brooks [1992] demonstrated that low stock prices, rather than small market values, move returns in January. Motivated by the strong linkage between size and financial difficulties, Chan and Chen [1991] isolated these two factors, and concluded that size proxies for distress in the size effect. This suggests that bankruptcy ought to be a factor in the economics of systematic risk.

Dichev [1998] investigated the connection between security returns and risk of bankruptcy, with the latter being measured by Ohlson's [1980] and by Altman's [1968] bankruptcy prediction models. In a test covering the period 1981 through 1995, he found that a portfolio composed of the firms most likely to become insolvent in the subsequent twelve month period consistently underperformed a combination of all risk portfolios. He also found that risk of bankruptcy is negatively associated with returns, confirmed that size has not been significantly associated with returns since the early 80's, and confirmed that book-to-market is positively associated with returns. The issue of whether bankruptcy risk is a component of systematic risk is the central question addressed by Dichev, but beta is not included in his tests, possibly due to the failure of beta to contribute even a partial explanation to cross-sectional variations in common stock returns, as shown by Fama and French [1992].

II. A Model of Risk and Expected Returns under Limited Liability

In this section I introduce a model of systematic risk and expected returns in an economy where businesses can fail, but where shareholders' liability is limited to their original investments. A firm is a one period business opportunity that requires an investment of cash at t_0, and generates an uncertain cash payoff $\tilde{X}$ one period later. At t_1 the payoff is entirely distributed to creditors and shareholders and the firm is liquidated. The firm may be leveraged. If $C \geq 0$ represents principal plus interest due at maturity, then the payoff to shareholders at t_1 will be either $\tilde{X} - C$ or zero, whichever is greater. Letting $\tilde{D}$ represent the uncertain payoff to shareholders, the payoff to the firm's creditors at t_1 will be the greater of $\tilde{X} - \tilde{D}$ or zero. Thus the owners of the firm declare bankruptcy when the firm's assets are insufficient to repay its creditors. Bankruptcy is presumed costless and there are no taxes.

Firms are inserted in an economy that includes a large number of other firms. Fractional shares of ownership in all firms are traded in a market that adheres to the assumptions underlying the Sharpe-Lintner-Mossin capital asset pricing model. I assume that there is a joint normal distribution between the firm's cash payoff $\widetilde{X}$ and the rate of return $\widetilde{r}$ on an index that includes all firms, with marginals $N(\mu,\sigma)$ and $N(\bar{r}_m, s_m)$ and correlation coefficient ρ. I assume that the density function $N(\bar{r}_m, s_m)$ is such that the probability of $\widetilde{r}_m < -1$ is immaterial.[2] However, the probability that the firm default in its obligations to creditors, i.e., that $\widetilde{X} < C$, can be anywhere in the open interval between 0 and 1.

The total market value of the firm's equity at time zero is denoted V_e and the rate of return on the firm's equity is $\widetilde{r}_e = \left(\widetilde{D}/V_e\right) - 1$. Systematic risk of equity is defined as usual to measure the sensitivity of rates of return on equity vis-à-vis rates of return on the market:

$$\beta^P = \frac{cov(\widetilde{r}_e, \widetilde{r}_m)}{var(\widetilde{r}_m)} \tag{1}$$

where the superscript "P" for beta indicates that this measure of beta allows for the possibility of bankruptcy.

If we combine the definitions of β^{P} and $\widetilde{r}_e$, it follows that:

$$\beta^P = \frac{1}{V_e} \frac{cov\left(Max\left(\widetilde{X} - C, 0\right), \widetilde{r}_m\right)}{var(\widetilde{r}_m)} \tag{2}$$

Default and bankruptcy occur when $\widetilde{X} < C$. If we let $\delta = (C - \mu)/\sigma$ be a standardized measure of leverage, then the probability of bankruptcy, denoted p, can be written as:

$$p = Pr\left\{\widetilde{X} < C\right\} = Pr\left\{\frac{X - \mu}{\sigma} < \delta\right\} = \int_{-\infty}^{\delta} \phi(t)dt = 1 - \pi(\delta) \tag{3}$$

where $\pi(\delta)$ is defined as the complement of the normal cumulative distribution function evaluated at δ, and $\phi(\cdot)$ is the standard normal density function. Note that p is a strictly increasing function of δ which is, in turn, strictly increasing with C.

For easier reference, here is a summary of the definitions adopted so far:

$\widetilde{X} \sim N(\mu, \sigma)$ - gross cash payoff to be generated by the firm at t_1

$\tilde{r} \sim N(\bar{r}_m, s_m)$ - market rate of return to be earned between t_0 and t_1

ρ - correlation between $\tilde{X}$ and $\tilde{r}$

C - amount due debtholders at t_1

$\delta = (C - \mu)/\sigma$ - standardized measure of leverage; the same as $\pi^{-1}(1 - Pr\{bankruptcy\})$

$\tilde{D} = Max(\tilde{X} - C, 0)$ - liquidating dividend at t_1

V_e - value of the firm's equity at t_0

$\tilde{r}_e = (\tilde{D}/V_e) - 1$ - rate of return on equity between t_0 and t_1

$\beta^P = cov(\tilde{r}_e, \tilde{r}_m)/var(\tilde{r}_m)$ - systematic risk of equity at t_0 when $p \in (0,1)$

$\phi(\cdot)$ - standard normal density function

$\pi(\delta) = \int_{\delta}^{\infty} \phi(t)\,dt$ - complement of the normal cumulative distribution function

$p = 1 - \pi(\delta)$ - probability of bankruptcy; $p \in (0,1)$

With these definitions and assumptions beta can be expressed in terms of parameters of the joint distribution of cash-flows and the market's rate of return, plus the value of equity. Starting from expression (2), and relying on results derived by Muthén [1990], it follows that (details in Appendix 1):

$$\beta^P = \frac{\rho \sigma \pi(\delta)}{V_e s_m} \tag{4}$$

Expression (4) holds regardless of how equity values are determined in this economy. As it stands, however, it does not allow us to establish a connection between systematic risk and fundamental characteristics of the firm without relying on value. More structure is needed so that we can reexpress value in terms of basic characteristics of the firm. In order to use the framework of the CAPM for this, it is necessary to check if the mean-variance approach to portfolio choice is suitable. Given the assumptions adopted so far, the only difference between security return distributions are their means and variances. Therefore, investors cannot be concerned with anything but means and variances when making portfolio decisions. However, because the mean and variance of truncated normal distributions are not independent, the fact that return distributions are specified by two parameters is not sufficient to justify mean-variance analysis.[3] In order to introduce the CAPM as an appropriate tool for computing V_e, I will add the assumption that investors have quadratic utilities. If investors have quadratic utilities then two-parameter distributions justify mean-variance analysis, despite the lack of independence between the mean and the variance. If the other assumptions of the Sharpe-Lintner-Mossin CAPM hold, the value of equity can be written as:

$$V_e = \frac{\overline{D} - \lambda\, cov\left(\widetilde{D}, \widetilde{r}_m\right)}{1 + r_f} \tag{5}$$

where r_f is the risk free rate, $\overline{D} = E\left(\widetilde{D}\right)$ is the expected value of the payoff to shareholders, and $\lambda = \left(\overline{r}_m - r_f\right)/s_m{}^2$ is the price of risk. Recognizing the effect of limited liability, the expected value of liquidating dividends is:[4]

$$\overline{D} = -\sigma\, h(\delta) \tag{6}$$

where $h(\delta) = \delta\, \pi(\delta) - \phi(\delta)$ is an auxiliary function such that $|h(\delta)|$ equals expected dividends for a firm with unit business risk. Since $\overline{D}$ must be positive under limited liability, (6) implies that $h(\delta) < 0$, which in turn implies that $\delta < \phi(\delta)/\pi(\delta)$. Figure 1 shows that expected dividends increase with business risk for any given level of debt and decrease with debt for any given level of business risk. The fact that anticipated dividends go up when business risk rises is due to the fact that shareholders can abandon assets to creditors whenever cash-flows are insufficient to meet debt obligations, implying that the increase in upside due to a higher σ is not matched by an increase in downside from the perspective of shareholders. This is, of course, the key to the analogy between options and common stock pointed out by Black and Scholes [1973].

Using expression (5), plus the definitions of λ and β^P produces the well known formula for value as the ratio of expected payoff to a risk adjusted discount factor:

$$V_e = \frac{\overline{D}}{1 + r_f + \beta^P\left(\overline{r}_m - r_f\right)} \tag{7}$$

Expressions (5) and (7) are standard CAPM results. Now we introduce a new one: an expression for equity beta in terms of two firm specific parameters (probability of bankruptcy and correlation of operating cash flows with the market) and three parameters that describe the economy (the risk-free rate, the expected return on the market and its standard deviation). This result is obtained by substituting the value of $\overline{D}$ given by (6) into (7):

$$\beta^P = \frac{-\rho\left(1+r_f\right)\pi(\delta)}{s_m\, H(\rho,\delta)} \tag{8}$$

where $H(\rho,\delta) = h(\delta) + s_m \lambda\, \rho\, \pi(\delta)$ is an auxiliary function such that $|H(\rho,\delta)|$ equals the value of a firm with unit business risk at a zero risk-free rate, as we will see below. Figure 2 shows how β^P behaves with respect to *p* for selected values of ρ. Although this is not apparent from the graph, β^P is not defined when *p*=0 or *p*=1. For any other probability of bankruptcy, $\rho = 0 \Rightarrow \beta^P=0$. In agreement with the analysis of the partial derivatives of β^P contained in Appendix 4, the graph shows that: (i) β^P increases (decreases) with *p* when ρ is positive (negative); and (ii)β^P increases with ρ whatever the likelihood of failure.

Combining (8) and (4) produces an expression for the value of equity:

$$V_e = \frac{-\sigma\, H(\rho,\delta)}{1+r_f} \tag{9}$$

Since equity values cannot be negative, expression (9) implies $H(\rho,\delta) < 0$, which is equivalent to $\overline{D} > \lambda\, cov\left(\tilde{E}, \tilde{r}_m\right)$, the first of Nielsen's [1992] sufficient conditions for positive asset prices.

An interesting consequence of imposing limited liability and allowing the possibility of failure to the CAPM's equilibrium pricing framework is that option pricing features and pricing features associated with portfolio diversification are at play simultaneously. Beta as a function of business risk (σ) has an inverted "*U*" shape, which in turn generates an upright "*U*" shape for equity value as a function of σ. The downward leg of value as a function of σ is associated with a decrease in the benefits of portfolio diversification (as the correlation with the market increases), whereas the upward leg is associated with the increasing value of the option to abandon.

Comparative statics results for β^P are a fruitful way of comparing the implications of a model that allows for bankruptcy in a CAPM economy, with behavior previously derived in theory or empirically observed. My results are in broad agreement with the literature, in the sense that for the more common values of the parameters the predicted signs are the same. However the implications are more complex, for instance, than those of the model proposed by Galai and Masulis [1976] which is based on the option pricing model. (See Appendix 4.)

The *unconditional* expected return on equity assumes the well known form:

$$\bar{r}_e = E(\tilde{r}_e) = r_f + \beta^P(\bar{r}_m - r_f) \tag{10}$$

with the measure of systematic risk reflecting the possibility of bankruptcy according to expression (8). A more difficult question concerns the expected return on equity *conditional* on the market's rate of return being, say $\hat{r}_m$. In the traditional view this would be written simply as the unconditional expectation, with $\hat{r}_m$ substituted for $\bar{r}_m$, or as $r_f + \beta^P(\hat{r}_m - r_f)$. However this does not hold when the probability of bankruptcy is nonzero and depends on observed rates of return on the market.[5] Starting from the definition of $E(r_e \mid \hat{r}_m)$ as $E(\tilde{D}/V_e - 1 \mid \hat{r}_m)$ we obtain (details in Appendix 2):

$$E(\tilde{r}_e \mid \hat{r}_m) = -1 + k\,(1 + r_f) + k\;\beta^P\;(\bar{r}_m - r_f) \tag{11}$$

where the factor $k = (1 - \rho^2)\,h(z)/h(\delta)$ is the ratio of conditional to unconditional expected dividends and z is a measure of risk analogous to δ but updated for the realized return on the market (full definitions are given in Appendix 2, or in the summary below.) From (11) we conclude that the expected return conditional on $r_m = \hat{r}_m$ has the following properties: (i) it is a nonlinear function of $\hat{r}_m$, ρ and p; (ii) because $lim_{\hat{r}_m \to -\infty} E(\tilde{r}_e \mid \hat{r}_m) = -1$, it naturally respects the lower bound of –1; and (iii) it is a linear function of the ratio of conditional to unconditional expected dividends and of that ratio times beta.[6]

Thus, although the *unconditional* expected return is still a linear function of systematic risk alone, the *conditional* expected return is a linear function of systematic risk multiplied by the adjustment factor k and of that factor by itself. The conclusion regarding $E(r_e)$ is similar to what was obtained in a non-bankruptcy model, but the conclusion regarding $E(r_e \mid \hat{r}_m)$ is not. This implies that after allowing for bankruptcy we should expect the time series average coefficient of $k\beta^{\mathrm{P}}$ (not just β^{P}) in cross-sectional regressions of returns above the risk-free rate on β^{P} to be $(\bar{r}_m - r_f)$.[7] The fact that $E(r_e \mid \hat{r}_m)$ is a nonlinear function of $\hat{r}_m$ that approaches the bankruptcy boundary of –1 asymptotically is in sharp contrast with the traditional model, where that relation is linear and a lower bound at –1 has to be imposed exogenously.

For easier reference, here is a summary of the latest definitions:

r_f - risk free rate

$\overline{D} = E(\widetilde{D})$ - expected value of the payoff to shareholders at t_1

$\lambda = (\overline{r}_m - r_f)/s_m{}^2$ - price of risk

$h(\delta) = \delta\, \pi(\delta) - \phi(\delta)$ - auxiliary function $h(\cdot)$; in absolute value equal to expected dividends when $\sigma = 1$

$H(\rho, \delta) = h(\delta) + s_m \lambda\, \rho\, \pi(\delta)$ - auxiliary function $H(\cdot)$; in absolute value equal to the value of the firm when $\sigma = 1$ and $r_f = 0$

$v = \rho(\hat{r}_m - \overline{r}_m)/s_m$ - standardized location measure for $\widetilde{X}$ given $\hat{r}_m$

$g = \sqrt{1-\rho^2}$ - standardized variability measure for $\widetilde{X}$ given $\hat{r}_m$

$z = (\delta - v)/g$ - the same as $\pi^{-1}(1 - Pr\{bankruptcy \mid \hat{r}_m\})$

$\overline{r}_e = r_f + \beta^P(\overline{r}_m - r_f)$ - unconditional expected return on equity

$k = (1-\rho^2)\, h(z)/h(\delta)$ - ratio of conditional to unconditional expected dividends, or $E(\widetilde{D} \mid \hat{r}_m)/E(\widetilde{D})$

The revised model of systematic risk and expected returns for an economy where businesses can fail is now complete. However, unless we can develop a measure of bankruptcy-adjusted beta, this model is not testable. The main purpose of the next section is to show how such a measure can be obtained from OLS beta.

III. The Relationship between Systematic Risk and OLS Beta

A. Systematic Risk and OLS Beta

As defined by equation (8), beta is an elusive measure, because it depends on the anticipated value of the covariance between $\widetilde{r}_e$ and $\widetilde{r}_m$. Financial analysts often rely on regression based methods to estimate beta. Popular books on value-based management, such as Young and O'Byrne [2001], describe the standard regression technique to calculate beta, while pointing out its limitations and providing highlights of the CAPM anomalies literature.[8] Another popular reference, Copeland, Koller and Murrin [1995], recommends the use of betas calculated by specialists such as Barra, Inc.[9], and suggests a rule for assessing beta when the specialists disagree.[10] A common approach is to combine ordinary least squares regressions with financial statement-based methods of estimating beta.[11] Research on the behavior of common stock returns often relies on OLS estimates of beta, albeit concentrating on portfolios, rather than on individual security betas.[12] For all these reasons it is important to investigate the relationship between OLS beta and the theoretical value of systematic risk. Furthermore, as we will see in this section, an estimator of bankruptcy-adjusted β^{P} can be based on this relationship.

Since the definition of systematic risk coincides with the slope of a regression of $\tilde{r}_e$ on $\tilde{r}_m$, then if $\tilde{r}_e$ and $\tilde{r}_m$ are generated according to the assumptions in Section II there can be but one reason for the expected value of OLS beta to diverge from β^{P} as given by expression (8): one of the requirements for OLS to be unbiased is violated. In this case the violated requirement is that the error term must be independently and identically distributed with mean zero. In an economy where returns to equity are bounded below at -1, even if a company is not delisted before $r_e = -1$ happens, only one such observation can ever be recorded. This means that the estimation of beta is based on samples where there is an equal chance of observing positive or negative disturbances around the expected value of $\tilde{r}_e$ when $\tilde{r}_m$ is high (assuming positive correlation with the market), but if the likelihood of failure is significant it is much more likely for us to observe positive rather than negative disturbances when $\tilde{r}_m$ is low. Intuitively the magnitude of the bias caused by the underrepresentation of very negative returns should be linked to the likelihood of distress and to the "tightness" of the relationship between $\tilde{r}_e$ and $\tilde{r}_m$: the bias should be less important when failure is remote (or linkage with the market is strong) rather than when failure is probable (or linkage is weak).

We can develop an expression for the OLS estimator of beta by conditioning the covariance in the numerator of (1) on the non-occurrence of default, as $\beta^{Z} = cov(\tilde{r}_e, \tilde{r}_m \mid \tilde{r}_e > -1)/var(\tilde{r}_m)$.[13] Here the superscript "Z" indicates that this is the standard OLS estimate of beta that ignores the possibility of bankruptcy. Using the definition of $\tilde{r}_e$, as we did to obtain expression (2):

$$\beta^Z = \frac{1}{V_e}\left[\frac{cov\left(\tilde{X},\tilde{r}_m \mid \tilde{X} > C\right)}{var\left(\tilde{r}_m\right)}\right] \tag{12}$$

From this result we obtain (details in Appendix 3):

$$\beta^Z = \frac{\rho\sigma}{V_e s_m}\left[\frac{\pi(\delta)^2 + \phi(\delta)h(\delta)}{\pi(\delta)^2 + \rho^2\phi(\delta)h(\delta)}\right] \tag{13}$$

Finally, using the CAPM formula for V_e, we reach an expression for β^Z in terms of ρ and *p*, plus the market's parameters:

$$\beta^Z = \left[\frac{-\rho\left(1+r_f\right)\pi(\delta)^2}{s_m\ H(\rho,\delta)}\right]\times\left[\frac{\pi(\delta)^2 + \phi(\delta)h(\delta)}{\pi(\delta)^2 + \rho^2\phi(\delta)h(\delta)}\right] \tag{14}$$

Figure 3 contains a plot of β^Z versus *p* for selected values of ρ, superimposed on a plot of β^P for the same values of ρ. Although this is not apparent from the graph, β^Z is not defined when *p*=0 or *p*=1. For any other probability of bankruptcy $\rho = 0 \Rightarrow \beta^Z = 0$. The graph shows that β^Z can overestimate or underestimate β^P, depending on the values of ρ and *p*. Comparing β^P (expression 4) with β^Z (expression 13), we observe that their ratio depends only on *p* and ρ (since δ is a function of *p* alone.) This suggests that we define a "bias function" $\Xi(\rho,p)$ as follows:

$$\Xi(\rho,p) = \frac{\beta^P}{\beta^Z} = \pi(\delta)\frac{\pi(\delta)^2 + \rho^2\phi(\delta)h(\delta)}{\pi(\delta)^2 + \phi(\delta)h(\delta)} \tag{15}$$

Figure 4 shows the behavior of Ξ with respect to ρ and *p*. The following properties hold:

(i) $\Xi(\rho, p)$ is symmetrical with respect to ρ, i.e. $\Xi(\rho, p) = \Xi(-\rho, p)$.

(ii) $\Xi(\rho, p)$ is continuous for $\rho \in [-1,+1]$ and $p \in (0,1)$.

(iii) The limit of Ξ as $p \to 0$ is 1^+, and the limit of Ξ as $p \to 1$ is 0^+. The maximum value of $\Xi(\rho, p)$ of approximately 1.426, occurs at $\rho = 0$, and $p \in (.345,.355)$

(iv) $\Xi(\rho, p)$ can be greater or less than 1, which implies that β^Z can underestimate or overestimate β^P.

(v) For any $\rho \in (-1, +1)$ there is a critical p^* such that: $\beta^Z < \beta^P$ for all $p \in (0, p^*)$; $\beta^Z = \beta^P$ for $p = p^*$; and $\beta^Z > \beta^P$ for all $p \in (p^*, 1)$.

(vi) For any $p \in (0, 1)$, Ξ is a decreasing function of ρ.

Having at our disposal expressions for the conditional and unconditional expected returns on equity and for the systematic risk of equity allowing or not for bankruptcy, we are prepared to discuss differences in expected returns according to the model proposed in this paper (henceforth denominated model "*P*", which allows for bankruptcy) and as traditionally computed (henceforth denominated model "*Z*", which ignores the possibility of bankruptcy.)[14] The ratio of unconditional expected returns above the risk free rate is:

$$\frac{E^P(\tilde{r}_e - r_f)}{E^Z(\tilde{r}_e - r_f)} = \frac{\beta^P}{\beta^Z} = \Xi(\rho, p) \tag{16}$$

Since in general $\Xi(\rho, p) \neq 1$, expression (16) implies that long-term averages of realized risk premiums according to models *P* and *Z* will diverge. When $\Xi(\rho, p) > 1$ an observer using model *Z* will be led to conclude that actual long-term average risk premiums are abnormally high, if risk premiums are in fact generated by model *P* (and vice-versa when $\Xi(\rho, p) < 1$.) If, in addition, companies with a given characteristic tend to cluster in a region of $p \times \rho$ where the bias is consistently up or down, observers using model *Z* will be led to conclude that the given characteristic is a relevant factor in explaining unconditional expected returns on equity, if risk premiums are in fact generated by model *P*.

From (11) the ratio of conditional expectations above r_f is:

$$\frac{E^P(r_e \mid \hat{r}_m) - r_f}{E^Z(r_e \mid \hat{r}_m) - r_f} = \frac{(k-1)(1+r_f) + k\ \beta^P\ (\bar{r}_m - r_f)}{\beta^Z(\hat{r}_m - r_f)} \tag{17}$$

As shown by numerical examples the ratio of conditional expected returns can be more or less than one whether $\beta^Z < \beta^P$ or $\beta^Z > \beta^P$. (See examples 1-3 below.)

The joint conclusions that OLS estimates of beta are biased and that expected return benchmarks must be adjusted to reflect the likelihood of bankruptcy raises important issues for research into the determinants of security returns. Since this research tends to focus on portfolio betas (rather than firm specific betas), it is important to stress that, although portfolio betas are less noisy than their firm specific counterparts, the portfolio technique is not capable of mitigating bias, if a consistent bias affects many companies in the same portfolio.[15]

B. An Estimator for Systematic Risk that admits Bankruptcy

By establishing a connection between β^{P} and β^{Z} expression (15) suggests a method for estimating systematic risk in an economy that admits bankruptcy, using OLS beta as the starting point. As we have seen, the ratio between β^{P} and β^{Z} depends only on the probability of bankruptcy and on the correlation between operating cash flows and returns on the market.

We can estimate a company's probability of bankruptcy by means of a model such as Ohlson's [1980], which generates a value for the probability of failure within a given time period as a function of certain financial statement based measures. Although the sample used by Ohlson is dated, the performance of his model continues to be good. (Dichev [1998] evaluates the performance of Ohlson's model in the period 1981-1995.)

Assessing the correlation of a company's operating cash flows with market returns is more difficult. Using figures drawn from annual statements of cash-flows is unlikely to produce a reliable and up-to-date proxy for ρ. A better alternative is to infer the value of ρ from expression (14) making use of the values previously obtained for β^{Z} and for p. That is, solve for ρ in $\hat{\beta}^{Z} = \beta^{Z}\left(\rho, \hat{\delta}\, ; \hat{r}_{f}, \hat{\bar{r}}_{m}, \hat{s}_{m}\right)$ given estimates for all variables under hats. This can be accomplished by means of numerical methods.[16]

An estimate of systematic risk of equity β^{P} is now obtained by substituting $\hat{\rho}$ and $\hat{p}$ into expression (8). Figure 5 shows ρ as a function of p and β^{Z}, with ρ being inferred as explained above. The graph indicates that: (i) ρ tends to 0 as $p \to 1$, and it does so faster the closer we are to p=1 and the higher the absolute value of β^{Z}; (ii) ρ tends to 1 as $p \to 0$, and it does so faster the closer we are to p=0 and the lower the absolute value of β^{Z}; (iii) there are combinations of p and β^{Z} values for which ρ does not exist. The convergence towards zero of correlations with the market as the likelihood of failure increases is economically appealing. It reflects, within the model's framework, the uncoupling of a firm's fortunes from market wide forces as the degree of financial distress increases. By causing a decrease in equity beta, this effect is largely responsible for the observation that portfolios of very distressed firms tend to underperform the market.

Figure 6 shows β^{P} as a function of p and β^{Z}. The plot reveals that: (i) β^{P} collapses towards zero as $p \to 1$; (ii) there are combinations of p and β^{Z} values for which β^{P} does not exist; (iii) the range of probabilities over which β^{Z} underestimates β^{P} shrinks as the absolute value of β^{Z} increases, and there is a value of β^{Z} above which OLS always overestimates β^{P}. The first two properties are evidently linked to the similar properties observed for ρ in the previous paragraph.

C. Further Discussion: Truncated Distributions and Missing Data

Two aspects of the argument which has led to the conclusion that OLS betas are biased warrant further discussion. One is the assumption that common stock returns follow truncated normal distributions. Another is the claim that catastrophic returns are underrepresented in databases of stock returns.

Regarding the assumption of truncated normal distributions for returns we begin with the fact that the normal distribution violates limited liability by having the entire real line as its support and becomes a less adequate representation of stock returns as the time interval over which returns are measured increases. Given the skewness of returns over longer periods a frequently used alternative is the lognormal, i.e., to assume that $log(1+\tilde{r}_e)$ is normally distributed. The lognormal has the advantages of being preserved under compounding, being positively skewed and not violating limited liability. However it does not capture other features of return distributions such as kurtosis.[17] But more serious, from the perspective of an economy where bankruptcy happens, is the fact that the lognormal stipulates $Pr\{\tilde{r}_e = -1\} = 0$. The model presented in this paper does not specify a distribution for common stock returns per se. Instead, after imposing a joint normal distribution on operating cash flows and market-wide returns and limiting the liability of shareholders to their original investment, it follows that the distribution of common stock returns is truncated normal at –1, with a probability mass at –1 equal to the probability of bankruptcy.[18]

Concerning the claim of underrepresentation of catastrophic returns we must begin by acknowledging that databases of common stock returns, such as CRSP, provide a wealth of information regarding the timing, the reason, and in many cases also the final delisting returns. Although, as Shumway [1997] and Shumway and Warther [1999] have demonstrated, actual return data on delisted firms is often missing, their papers suggest ways of correcting for this problem. The important point regarding the possibility of a bias in OLS beta is that *even if full delisting return information was available*, the bias would still exist. The reason is that all we can observe about any company's history is a unique realization, and the derived time series of common stock returns is but one possible outcome of a stochastic process with unknown parameters. Furthermore, bankruptcy is an absorbing state: there can be but one outcome with a return of –1, and it is this that leads to the difficulty of capturing the impact of bankruptcy on systematic risk.

The model of systematic risk under limited liability introduced in Section II makes up for this scarcity of data with distribution shapes and other assumptions about the economy. It is worth stressing that the tobit estimator,[19] which deals with censored or truncated data, is not appropriate for the problem at hand. The problem we face here is not one where the data are truncated, but one where there is an absorbing state.

D. Examples

In order to help develop an intuitive basis for the model, let us now specify values for the parameters and observe how systematic risk and expected returns behave in three special cases. The examples below were solved with the market parameters fixed at: $r_m = .15$, $s_m = .23$ and $r_f = .05$.

- Example 1 (Intermediate p and Relatively High ρ): Let p=25.2 % and ρ=60 %. Systematic risk values according to the model are β^Z = 2.82 and β^P = 3.29, with a ratio Ξ = 1.168. Figure 7 shows that conditional expected returns using model *P* (bankruptcy-adjusted) are less than those obtained using model *Z* (the traditional view) in the range of approximately $\hat{r}_m \in (-.25, +.25)$, being higher outside that range. That is, an analyst computing expected returns in the traditional way would alternate between thinking that actual returns are too high and too low depending on the value of $\hat{r}_m$, if returns are actually generated according to model *P*. The value chosen for *p* of 25.2 % is at the high end of failure probabilities during the period 1973-1980 when returns for the most distressed firms were not found to be less than average market returns, which is consistent with Figure 7.

- Example 2 (High *p* and Relatively Low *ρ*): Now let *p*=90.9 % and *ρ*=20 %. Systematic risk values according to the model are β^Z = 4.50 and β^P = 2.42, with a ratio Ξ = .537. Figure 8 shows that conditional expected returns according to the model exceed traditional expected returns for all $\hat{r}_m$ above zero. That is, an analyst computing expected returns using model *Z* would believe that actual returns are too low when $\hat{r}_m > 0$, and too high when $\hat{r}_m < 0$, if returns are actually generated according to model *P*. The value chosen for *p* of 90.9 % is at the high end of probabilities of failure during the period 1981-1998 when returns for the most distressed firms were found by Dichev [1998] to be less than average market returns. Thus the graph is consistent with bankruptcy's risk-return paradox, during a period when the average market rate of return was positive.

- Example 3 (Low *p* and Intermediate *ρ*): For the last example let *p*=1 % and *ρ*=40 %. Systematic risk values according to the model are β^Z = .80 and β^P = .84, with a ratio Ξ = 1.046. The value set for *p* is associated with the larger and safest firms and the correlation coefficient is similar to the average for the NYSE and AMEX from 1973 until 1998. Figure 9 shows that conditional expected returns according to the model approximate traditional expected returns, being just slightly above when $\hat{r}_m$ is positive. An analyst computing expected returns would not find significant discrepancies between actual values and the traditional benchmark.

The first example has a lower probability of bankruptcy and a higher correlation with the market than the second. In the first $\beta^Z < \beta^P$ and in the second $\beta^Z > \beta^P$. The third example (with a low level of bankruptcy risk and an average value for the correlation coefficient) has $\beta^Z \approx \beta^P$.

We observe in Example 2 that expected returns according to the higher of the two betas can be less in absolute value than expected returns according to the lower beta over a wide range of returns due to the nonlinearity of expected returns with respect to r_m. The three examples and their associated plots of r_e versus r_m show how *p* and ρ interact to generate a bias in β^Z. They also illustrate the need to make adjustments to expected returns when the risk of bankruptcy is significant.

IV. Test Design and Data Collection

A. Testing Procedures

The basic question that motivates this research is whether the introduction of bankruptcy risk into the standard framework of the CAPM can explain the paradoxical behavior of stock returns observed by Dichev[1998]. The model which we developed in Section II to address this issue (model *P*) has produced two expressions for common stock returns which are testable: one for unconditional expected returns (10) and another for conditional expected returns (11). Both expressions diverge from their counterparts in a version of the CAPM that does not allow for bankruptcy (model *Z*). Both expressions require adjusted measures of systematic risk derived in Section III, which are different from the OLS measures traditionally employed. Thus, tests of whether the inclusion of bankruptcy risk adds explanatory power can be formulated in terms of whether expressions (10) and (11) are in better agreement with the behavior of actual returns than their equivalents in the traditional framework.

Two frequently used methods to test models of expected returns -- which were employed by Dichev [1998] -- are the “sorted portfolios” (SP) and the “cross-sectional regressions” (CSR) methods. The first method is analogous to an analysis of variance in which: (i) firms are sorted according to the value of one or more explanatory variables *x*; (ii) firms are assigned to, say, ten portfolios either by deciles or respecting a set of fixed cutoff points for *x*; (iii) average returns are computed for each portfolio; (iv) the pattern of portfolio returns is observed and we test whether the association between *x* and returns is in agreement with theory. The advantage of the SP approach is that the pattern of returns with respect to the portfolio formation variables is not required to be linear; the disadvantage is that the explanatory variables must be discrete and few in number (typically one or two.) The CSR method involves: (i) estimating the coefficients of periodic cross-sectional regressions on a set of explanatory variables; (ii) computing averages and standard errors of these coefficients over time; and (iii) testing whether the sign and/or value of the average coefficients are in agreement with theory. The advantage of the CSR approach is that any number of discrete or continuous explanatory variables can be easily included in the test; the disadvantage is that a linear relationship is assumed.

We implement SP by forming ten portfolios at the beginning of each month, from January 1973 until January 1999. With the variable of interest being *probability of bankruptcy* we allocate firms in the lowest decile by probability of bankruptcy to portfolio 1, those in the second decile to portfolio 2, and so on. Then, for each month we compute averages over all firms in each portfolio of the following variables: (a) realized compound return during the following twelve month period; (b) OLS beta; (c) bankruptcy adjusted beta; (d) correlation coefficient obtained as described in Section III; (e) expected returns without adjusting for bankruptcy; (f) conditional expected returns adjusted for bankruptcy; (g) size; (h) book-to-market equity ratio; and (i) probability of bankruptcy.

We implement CSR from January 1973 until January 1999, by running a set of cross-sectional regressions in which the dependent variable is the realized compound return during the following twelve month period, and the set of independent variables includes variables which are supposed to move common stock returns according to the traditional or to the bankruptcy adjusted models, plus other factors that have been found to be associated with returns, such as size, book-to-market equity, and probability of bankruptcy.

Regarding the traditional model *Z*, the regressions take the form:

$$r_{i,t} = a_{0,t} + a_{1,t}\beta^{Z}_{i,t} + a_{3,t}S_{i,t} + a_{4,t}BME_{i,t} + a_{5,t}P_{i,t} + u_{i,t} \tag{18}$$

where the subscript *i* indicates a particular firm, the subscript *t* refers to time, $S_{i,t}$ is size, $BME_{i,t}$ is book-to-market equity, and $P_{i,t}$ stands for probability of bankruptcy. According to theory the time series average of the intercept $a_{0,t}$ should be the risk-free rate, the time series average of

$a_{1,t}$ should be $(\bar{r}_m - r_f)$ and the time series averages of the other coefficients should be statistically indistinguishable from zero. However, from an extensive literature on this family of regression models, we expect that the time series average of the coefficient on size will be negative, on book-to-market equity will be positive and on beta will be indistinguishable from zero. From Dichev [1998] we also expect that the time series average of the coefficient on probability of bankruptcy will be negative.

Regarding the positive risk of bankruptcy model *P*, the regressions are:

$$r_{i,t} = b_{0,t} + b_{1,t} K^*_{i,t} + b_{2,t} B^*_{i,t} + b_{3,t} S_{i,t} + b_{4,t} BME_{i,t} + b_{5,t} P_{i,t} + v_{i,t} \qquad (19)$$

where $K^*_{i,t}$ is the expected dividend revision factor given the return on the market, i.e.,

$$K^*_{i,t} = \left(1 - \rho^2\right) \frac{h(z)}{h(\delta)} \qquad \text{for company } i\text{, at time } t$$

and $B^*_{i,t}$ is revised beta, or beta multiplied by $K^*_{i,t}$:

$$B^*_{i,t} = K^*_{i,t}\ \beta^P \qquad \text{for company } i\text{, at time } t$$

According to expression (11) the time series average of $a_{0,t}$ should be -1, the time series average of $a_{1,t}$ should $(1 + r_f)$ and the time series average of $a_{2,t}$ should be $(\bar{r}_m - r_f)$. My expectation regarding the coefficients on *S*, *BME*, and *P* is that they should be zero according to expression (11). However, if model *P* is an improvement on model *Z*, I expect simply that the variables included in regression (19) -- K^* and B^* -- will be relatively more important than β^Z is in regression (18), even if extra-CAPM factors continue to play a role.

B. The Sample and Variable Definitions

The sample includes all industrial firms with monthly return data on CRSP and annual financial statement data on COMPUSTAT, with SIC codes 1 through 3999 or 5000 through 5999, incorporated in the United States, with common stock listed by the NYSE, AMEX or Nasdaq. Tests span 313 months from January 1973 until January 1999. Although complete data to estimate bankruptcy probabilities becomes available for some companies as of January 1972, I have chosen to drop 1972 from the test period because the effective sample size grows from 89 to over 1,100 companies during that year. The test period ends in January 1999 -- the last full year of currently available CRSP returns – because of the need to compute one-year ahead returns for each month of the report period. Monthly return data was collected from CRSP as of January 1968 so that OLS betas estimated over the 60 month period preceeding the time for which they are assumed to measure systematic risk could be computed for January 1973.

Probability of bankruptcy is estimated by means of Ohlson's [1981] model #1 that predicts bankruptcy within one year. The model is:[20]

$$\begin{cases} JO = -1.32 - .407 \times lnTA + 6.03 \times TLTA - 1.43 \times WCTA + .0757 \times CLCA - 2.37 \times NITA \\ \qquad -1.83 \times FUTL + .285 \times INTWO - 1.72 \times OENEG - .521 \times CHIN \\ \\ P = \left(1 + Exp\left(-JO\right)\right)^{-1} \end{cases}$$

where the variables are defined as follows: *lnTA* is the log of total assets divided by the GNP price level index (total assets from COMPUSTAT item # 6)[21]; *TLTA* is total liabilities divided by total assets (total liabilities from item # 181); *WCTA* is the ratio of current assets less current liabilities over total assets (current assets from # 4 and current liabilities from # 5); *CLCA* is the current ratio; *NITA* is net income divided by total assets (net income from # 172); *FUTL* is funds provided by operations divided by total liabilities (funds from item # 110 when available; otherwise estimated as cash from operations adjusted for changes in working capital)[22]; *INTWO* is one if net income was negative for the last two years, otherwise zero; *OENEG* is one if total liabilities exceeds total assets, otherwise zero; *CHIN* is the ratio of year on year change in net income over the aggreagate absolute value of the previous two years net income figures; finally *P* is the estimated probability of bankruptcy over the following twelve month period. All variables needed to compute *P* are assumed to be known at fiscal year end.

Book value of common equity (or just book equity) at the end of fiscal year *y* is the total reported accounting value of common stock, capital surplus, and retained earnings, minus stock held in treasury at the end of fiscal *y* (item # 60). Book equity *known* at the beginning of month *t*, and therefore available for portfolio formation at that time, is book equity at the end of the most recent known fiscal year. A publication delay of six months is assumed from the end of any company's fiscal year until the time when financial statement information for that fiscal year is known.

Market value of common equity (or just market equity) at fiscal year *y* is obtained by multiplying the number of shares outstanding (item # 25) by the closing share price, both measured at the end of fiscal year *y* (item # 199). Size at fiscal year *y* is defined as log of market equity at *y* in millions of dollars. Size known at the beginning of month *t*, is size at the end of the most recent known fiscal year. Although information on the market value of common equity is available to investors practically continuously, we assume that size is known with a delay of six months to be consistent with Dichev [1998]. Book-to-market equity (or just book-to-market) known at the beginning of month *t*, is the ratio of book equity to market equity, both known at that point in time.

Monthly returns include delisting returns provided by CRSP. When delisting returns are not available in CRSP, Shumway's [1997] estimate of –30% for firms listed by the NYSE and AMEX, and Shumway and Warther's [1999] estimate of –55% for firms listed by the Nasdaq, were substituted for the missing returns. Year ahead returns are monthly returns compounded from the beginning of the portfolio formation month until the firm is delisted or until the end of the twelve month period, whichever occurs first.

Beta at the beginning of month *t* is estimated as the sum of the coefficients of current and one month lagged $\left(\hat{r}_m - r_f\right)$ of a regression with $\left(r_e - r_f\right)$ as the dependent variable, following the procedure recommended by Dimson [1979] to address the problem of infrequent trading. I require that at least 24 return observations (consecutive or not) be available for beta to be computed. Market returns are assumed to be those of an equally-weighted portfolio of all securities included in the sample, whether or not they are included in the final tests. The risk-free rate is the return on CRSP's one year bond (indno=1000706).

The coefficient of correlation (ρ) between operating cash flows and market returns is inferred by means of a numerical procedure (described in Section III-B) that finds the value of ρ that solves the equation obtained by substituting p and OLS beta into expression (14). Given measures of ρ and p, expression (8) is then used to produce an estimate of bankruptcy-adjusted beta. Expected returns according to model *Z* are generated by means of the standard CAPM formula with firm-specific OLS betas. Expected returns according to model *P* are computed with expression (11).

Unlike Dichev [1998], none of the data items used in my the tests is winsorized or trimmed. Dichev uses trimming -- which he applies to all variables except market capitalization -- as a protection against having a few extreme observations drive his results.[23] In this paper trimming has been rejected for several reasons. The first reason is that trimming returns does not affect just the realized year-ahead return statistics, but also estimates of OLS beta, which in turn is the critical input for bankruptcy-adjusted beta and for expected returns according to models *Z* and *P*. The second reason is that Ohlson's [1980] model was derived with untrimmed data, and that not only a ranking by probabilities, but the actual estimate of the probability of bankruptcy is required to compute betas and expected returns. The first and second reasons involve the issue of when to start and when to stop trimming when variables are used in several steps to derive further variables, which brings us to the third reason: a portfolio of very distressed firms is by construction extreme in terms of returns, probabilities of bankruptcy and systematic risk, which means that trimming these variables at, say, their aggregate 1st and 99th percentiles affects that portfolio more than any other. The fourth reason is that Dichev's paradox is present with (as Dichev demonstrates) or without trimming (as I demonstrate.)

C. Descriptive Statistics

Summary statistics for size, book-to-market, probability of failure and monthly returns are given in Table I by time period and stock exchange listing. Firm size increases over time and firms listed in the NYSE-AMEX are larger than those listed in the Nasdaq. Book-to-market displays a significant downward trend over time for both exchange sub-samples, but the average *BME* values are similar across exchanges at each decade. Probability of failure increases over time and is significantly higher for the Nasdaq than for the NYSE-AMEX during each of the three decades. We observe that in both exchange sub-samples the mean probability of bankruptcy more than doubles from the 70's to the 90's, and the overall mean practically trebles due to the growing participation of Nasdaq companies in the sample. Also as expected, monthly returns for the Nasdaq show more volatility than those of the NYSE-AMEX, and volatility for the Nasdaq has increased over time.

Table II provides summary statistics for traditional OLS betas, bankruptcy-adjusted betas, the coefficient of correlation with the market and for OLS beta F-tests, by time period and stock exchange listing. Both β^{Z} and β^{P} have lower means and lower standard deviations in the NYSE-AMEX than in the Nasdaq. The means of both measures of beta display a negative trend in the NYSE-AMEX over time, which is compensated by a positive trend in the Nasdaq in the case of β^{Z}, while the standard deviations display a positive trend in all exchanges. The mean and standard deviation of the correlation between cash flows and the market appear to be similar across exchanges, but there is a downward trend in the mean and an upward trend in the standard deviation from 1973 until 1998. As indicated by the F-test p-values for the OLS regressions that generated estimates for β^{Z}, there is considerably less precision in the estimates of beta for the Nasdaq than in those for the NYSE-AMEX, but the degree of precision has been getting worse over time for all exchanges. It is also true (results not shown) that the F-test p-values are uniformly higher for the most distressed firms' portfolios.

A strong message coming from Tables I and II is that firm-specific risk measures have generally increased over the past three decades: both the probability of bankruptcy and the volatility of returns have gone up. As we have seen in Section II systematic risk is a function of probability of bankruptcy and correlation with the market. Since economy wide systematic risk must be one, the increase in p had to be matched by a decrease in ρ, which we indeed observe. Another message is related to the joint observations that F-test p-values have been increasing over time and that they are highest for the most distressed firms. If we combine this with the fact that bankruptcy prediction models are most precise for the most distressed firms we conclude that there is a mechanism of uncertainty that makes it most difficult to measure beta exactly where it is easiest to measure probability of bankruptcy, and vice-versa.

V. Test Results

A. Sorted Portfolios Method

Results of the SP method are summarized in Tables III, IV and V. Results are presented separately for firms listed on NYSE-AMEX and on Nasdaq. Results are broken down further over two time periods: January 1973 to December 1980 (early) and January 1981 to January 1999 (recent). The reason for splitting the analysis in this fashion is that these time periods and exhanges display strong differences in terms of levels of bankruptcy risk which serve to illustrate the relative performances of models *Z* and *P*. Dichev [1998] presented detailed results according to the exchange listing breakdown, and mentioned that the risk-return puzzle was not present during the early period when probabilities of bankruptcy were lower.

The behavior of *BME* vis-à-vis probability of bankruptcy is in line with previous research. During the early period for the Nasdaq, and during the recent period for the NYSE-AMEX and the Nasdaq, *BME* starts growing along with *p*, but drops again as we move into the highest probability portfolios. During the early period for the NYSE-AMEX the trend in *BME* never turns down, but for these exchanges the most distressed portfolio had an average risk of failure of only 23.49 %. (Refer to tables IV and V, both panels.) Dichev [1998] explains the reason for lower *BME*'s for the most distressed firms, and also points out that distress probably cannot account for the book-to-market effect as follows:

> "The reason is that, unlike market value, book value of the most distressed firms is often completely wiped out by losses or is even negative. Thus, even if firms with high bankruptcy risk have higher returns, the nonmonotonic relation between bankruptcy risk and book-to-market suggests that a distress explanation is unlikely to account for the book-to-market effect."[24]

The behavior of expected returns with respect to risk of bankruptcy predicted by the model developed in Section II (model *P*) suggests a continuation to the argument above: the trend in *BME* turns negative at the highest *p*'s because book value is often wiped out by losses *and* because there is a point after which further increases in *p* have a positive impact on market value by means of the diversification and option-to-abandon mechanisms. There are forces acting on both numerator and denominator causing *BME* to fall. The increase in market value is just another aspect of the observed decrease in those firms' year-ahead returns: according to model *P* nonmonotonic relations between *p* and *BME* and between *p* and year-ahead returns are inseparable, being triggered only at the highest levels of bankruptcy risk.

Size has a marked tendency to fall as risk of bankruptcy increases for all exchange/period combinations. This fact, combined with very low returns for the most distressed firms, does not necessarily imply that bankruptcy risk cannot help to account for the size effect. It does mean, however, that if distress is related to the size effect, then the size effect must happen for relatively healthy small firms. This serves to underscore the importance of high quality bankruptcy prediction models in this line of research. The problem is that, as Ohlson [1980] and Shumway [2001] have noted, the performance of bankruptcy prediction models decays rapidly as firms become healthier.

Beta estimated by ordinary least squares (β^{z}) displays a tendency to increase across all exchange/period combinations. This observation has intriguing implications concerning the performance of OLS beta in cross-sectional models of returns. First, this means that the standard CAPM with OLS betas cannot even begin to account for below average yields for the most distressed firms. Another consequence is that it should not be surprising to find statistically insignificant or even significantly negative coefficients for OLS beta in cross-sectional regressions of returns when bankruptcy risk is very high! This can happen even if OLS beta is in fact linearly related to common stock returns of healthier firms.

The behavior of beta adjusted for bankruptcy (β^{p}) is what we would expect in light of the discussion contained in Sections II and III. For [NYSE-AMEX, early] with top portfolio $p=23.49$ %, we have top portfolio $\beta^{z} = 1.27$ and $\beta^{p} = 1.44$, reflecting the downward bias in OLS beta within this range of probabilities (see Figures 4 and 6). Results for [Nasdaq, early] are very similar. For [NYSE-AMEX, late] with top $p=45.58$ % the degree of underestimation by OLS diminishes and so does the gap between the two betas, with $\beta^{z} = 1.11$ and $\beta^{p} = 1.15$. The most interesting outcome happens for [Nasdaq, late], where top *p* reaches 90.53 %, and the direction of the trend in beta splits in two: OLS beta continues to rise briskly with *p*, but bankruptcy-adjusted beta drops sharply: $\beta^{z} = 1.42$ and $\beta^{p} = .58$.

The correlation coefficient displays an overall tendency to fall with p, and the rate of decrease tends to be faster during periods of high p. For [NYSE-AMEX, early] the correlation coefficient falls by 13.4 percentage points, whereas for [NYSE-AMEX, late] the drop in ρ reaches 26.1 points and for for [Nasdaq, early] 31.1 points. The most striking decrease happens for [Nasdaq, late], where ρ goes from 55.3% to only 5.6%, a decrease of 49.7 points! All of this is consistent – as it must be -- with Figure 5 and with expression (14) by means of which the correlation coefficient was obtained. Necessarily consistent, but not devoid of economic significance. The correlation between the Nasdaq firms in portfolio #10 and the market drops to such low levels in the more recent period only because OLS betas did not increase enough as the mean risk of bankruptcy for that portfolio shot upwards to 90.53%.

Actual year-ahead returns for the most distressed portfolio in the [NYSE-AMEX, late] sub-sample are above average, which is not what Dichev reports. This discrepancy is due to the fact that I have not trimmed returns at their 1st and 99th percentiles.[25] (My reasons for not trimming are given in the previous section.) That this does not eliminate the paradox is clear from the results in panels A and B of Table V, where actual year-ahead returns for the Nasdaq are respectively below and far below average. The mean twelve month yield for the most distressed firms was only 3.9% from 1981 until 1998, a period when the mean yield for all portfolios reached 13.7%. If we compare the average risks of bankruptcy for portfolio #10 for each exchange/period combination, there is a strong indication that the actual *level* of probability of failure, and not just the portfolio classification, is the critical factor in the observed pattern of returns. This was pointed out by Dichev, who stated that "an explanation for why the relation between bankruptcy risk and returns is different before and after 1980 is likely related to the fact that (...) bankruptcy risk was comparatively low before 1980, and increased substantially thereafter."[26]

The expected return data in the final columns of Tables III, IV and V compared with each other, and with the actual return patterns discussed in the previous paragraph, test the central argument of this paper. A graphical display of actual and expected returns according to models Z and P is given in Figures 10, 11 and 12. Let us look first at the early period. Regarding the

NYSE-AMEX (Figure 11) the three return series – actual, *Z* and *P* – show a tendency to increase with *p*. Regarding the Nasdaq (Figure 12) actual returns display no clear trend, *Z* returns have a positive trend, and *P* returns are up except for a slight decrease at portfolio #10. In the early period the traditional and the bankruptcy-adjusted models have a similar performance: both reflect the trend in the NYSE-AMEX case and neither reflects the mixed pattern in the Nasdaq case. The performance of the two models for the combination of all exchanges is contrasted in Figure 10, which is clearly dominated by behavior in the NYSE-AMEX sub-sample.

Now for the more recent period. Regarding the NYSE-AMEX (Figure 11), the actual, Z and *P* return series display a small positive trend from portfolio #1 to portfolio #9. From that point to portfolio #10 actual returns drop, in which they are followed by model *P*, but not by model *Z* according to which returns continue to go up. Regarding the Nasdaq, the actual, Z and *P* return series display a small positive trend from portfolio #1 to portfolio #5. From that point to portfolio #10 actual returns drop sharply, *P* returns remain stable until portfolio #9 and then drop sharply, and *Z* returns continue to go up at an even faster pace. In the more recent period the traditional and the bankruptcy-adjusted models have a similar performance for the lower risk of failure portfolios. The trend in returns generated by the traditional model for the riskier portfolios is opposite the trend displayed by actual returns. However the trend generated by the bankruptcy-adjusted model is capable of tracking – albeit in attenuated fashion – the steep decline displayed by actual returns. Both the failure of model *Z*, and the relative success of model *P* in generating return expectations can be attributed largely to the different ways in which these models measure systematic risk. The other important difference between the two models is that only model *P* captures the value of shareholders' option to abandon.

In summary, throughout the period from 1973 until 1980, when risk of failure was lower and Dichev's paradox was not apparent, model's *Z* and *P* are indistinguishable in their ability to predict rate of return patterns with respect to bankruptcy risk portfolios. However, from 1981 until 1998, when risk of failure was higher and the full fledged paradox was observable, model *Z* not only fails to anticipate the reversion in the trend of rates of return, but actually predicts an even steeper rate of increase across the riskier portfolios. At the same time model *P* anticipates the reversion in trend, tries to keep up with actual returns, and in so doing opens up a large gap vis-à-vis the traditional model. The plots on the left (for the early period) and on the right (for the more recent period) of Figure 10 illustrate these results and conclusions for the NYSE, AMEX and Nasdaq combined.

B. Cross-Sectional Regressions Method

Results of the CSR method are summarized in Table VI, where panels A and B provide coefficient estimates and *t*-statistics for the traditional (*Z*) and bankruptcy-adjusted (*P*) versions of the model. Regarding model *Z*, the coefficients on *S*, *BME*, and *P* are, as expected, significantly negative, positive and negative. The *t*-statistics are larger than those reported by Dichev because the independent variable here consists of twelve-month-ahead returns per dollar invested, rather than monthly returns. The coefficient on β^Z, which according to theory should be positive, is significantly negative instead. This is due to the fact that OLS betas rise monotonically with the probability of failure, but returns drop precipitously for the most distressed firms. Other than the magnitude of the *t*-statistics, the only divergence with Dichev's results in terms of these regressions is the finding the size effect is still active from 1981 until 1998, although being weaker recently than it was between 1973 and 1980. (Separate regressions were run for the early and recent time periods, but these results are not presented here.)

As regards model *P* the conclusion for *S*, *BME*, and *P* is the same as above, except that no expectations existed for this version of the model, other than the theoretical one of zero explanatory power for any combination of extra-CAPM variables in the presence of the K^* and B^* constructs. Turning our attention to these constructs we can reach two conclusions: the first is that the coefficient on the ratio of conditional to unconditional expected dividends (K^*) is significantly positive for all regressions estimated, but smaller in magnitude than anticipated; the second conclusion is that the coefficient on revised systematic risk (B^*) is significantly positive in the absence of any variable other than K^* or *BME*. We should also note that the constant term is too large and the coefficients on K^* and B^* are too small compared with what the model predicts, even in regression P1. Perhaps the most interesting contrast between the traditional (*Z*) and bankruptcy-adjusted (*P*) versions of the model is that in the former systematic risk behaves in a manner which is opposite of what theory predicts, but in the latter both the ratio of expected dividends and revised systematic risk behave as suggested by theory, unless other variables are added to the model.

What accounts for the loss of significance of B^* in regressions P2 and P5 and for the sign reversion in P6? We must start by acknowledging that, no matter how significant the theoretical constructs may be in a bankruptcy-adjusted version of the CAPM, the continued significance of size and book-to-market remains a puzzle. The relevance of probability of bankruptcy and the negative sign on B^* in regression P6 are due to a problem of collinearity, i.e., to the strong association between these two variables under model P: B^* is a function of probability of bankruptcy through K^* and β^P. As for the diminished importance of B^* in regressions P2 and P5, a likely cause -- which deserves to be pursued in future research -- is the very high error rate of the bankruptcy prediction model in the range of low to moderate likelihoods of failure. The joint fact that small firms are more likely to fail, distressed firms are expected to produce low yields, but healthy small firms are expected to produce high yields is beyond the discriminating ability of the bankruptcy prediction model employed, and this creates a role for factors such as size and book-to-market equity in the cross-sectional regressions.

VI. Conclusion

The recognition that companies may fail to meet their obligations to creditors and that bankruptcy is possible implies that certain long held beliefs about systematic risk and expected returns must be revised, even under the assumptions of a standard single-period CAPM economy. According to the model presented in this paper the OLS estimator of beta is biased, sometimes severely so, being either above or below target depending on the likelihood of bankruptcy and on the degree of correlation between the firm's cash flows and market performance. Consequently, because unconditional expected returns are proportional to beta, according to both the bankruptcy-adjusted model and the traditional view, observers who disregard the possibility of failure will often detect significant deviations between long-term average returns and their own benchmarks. An even more surprising reversal of beliefs occurs regarding expected returns *conditional* on the realized rate of return on the market: whereas traditionally this has been computed as in the unconditional case, by substituting the observed return on the market for its long term expectation, according to the model that admits bankruptcy these returns are a linear function of the ratio of conditional to unconditional expected dividends, and of that ratio times beta.

Results of empirical tests which cover a period extending from January 1973 until January 1999, based on a sample of industrial companies incorporated in the United States and listed in the NYSE, AMEX or Nasdaq, support the model's predictions. From 1973 until 1980, when risk of failure was relatively low and Dichev's paradox was not apparent, the traditional and bankruptcy-adjusted models are indistinguishable in their ability to predict rate of return patterns with respect to the distress portfolios. However, from 1981 until 1998, when risk of failure was relatively high and the paradox was observable, the traditional model fails to anticipate the reversion in the trend of rates of return and predicts instead an even steeper rate of increase for the riskier portfolios. At the same time the bankruptcy-adjusted model anticipates the reversion in trend and in attempting to keep up with actual returns opens up a large gap with respect to the traditional model.

In sharp contrast with the traditional model, we find that coefficients on the ratio of expected dividends and on revised systematic risk in cross-sectional regressions of year-ahead returns on those two variables are, as predicted, significantly positive. However size and book-to-market equity continue to exert significant negative and positive influence on returns, whether the variables suggested by the bankruptcy-adjusted model are included in the regressions or not.

The explanation offered to Dichev's [1998] finding that portfolios of very distressed firms underperformed the market between 1981 and 1995 is that, as the probability of failure increases, two factors operate simultaneously to lower expected returns. The first is a significant drop in the linkage between the firm's cash flows and the economy at large, which affects returns via a reduction in systematic risk. The second is the diminished relevance of downside risk to investors, which affects returns by increasing the option value of equity. Thus, sub-average returns to very distressed firms are entirely consistent with equilibrium prices in an efficient capital market.

Is the risk of bankruptcy a systematic risk? If we interpret the question to be whether risk of bankruptcy and systematic risk are related, then the answer suggested by the model and by the empirical tests described in this paper is affirmative. However, if we interpret the question to be whether systematic risk *increases* with the risk of bankruptcy, then it depends. In theory, if the risk of bankruptcy could change without a concurrent change in the correlation between the firm's operational cash flows and the market, then the answer would once more be affirmative. But in practice we have found that, as the probability of bankruptcy approaches one, there is a simultaneous decrease in the correlation between cash flows and the market which is strong enough to produce a sharp decrease in systematic risk and in expected returns. A clear inference that can be drawn from this research is that correlation with the market and risk of bankruptcy are as inextricably associated in practice, as the diversification and option-to-abandon components of firm value are in theory.

These conclusions are relevant to the ongoing debate on the relative importance of beta, size, book-to-market equity and other factors in explaining variations in common stock returns. They demonstrate that a model that admits bankruptcy helps to reconcile the CAPM with the finding that extra systematic risk variables are associated with returns, and in so doing suggest that incorporating additional factors that potentially distort covariances and expectations -- such as taxes and costly bankruptcy – holds the promise to sharpen our understanding of common stock returns.

Appendix 1: An expression for equity beta

We begin with expression (2) from Section II, adding the definition of an index variable $\tilde{I}$ that takes the value 1 if $\tilde{X} > C$ and the value 0 if $\tilde{X} \le C$:

$$\beta^P = \frac{cov\left(Max\left(\tilde{X}-C,0\right),\tilde{r}_m\right)}{V_e\, var\left(\tilde{r}_m\right)} = \frac{cov\left(\left(\tilde{X}-C\right)\tilde{I}\,,\tilde{r}_m\right)}{V_e\, s_m^2} = \frac{Q'}{V_e\, s_m^2}$$

The term Q' can be rewritten as:

$$Q' = cov\left(\tilde{X}\,\tilde{I}\,,\tilde{r}_m\right) - C\,cov\left(\tilde{I}\,,\tilde{r}_m\right)$$

In order to use the results in Muthén [1990] – which were derived for standard normal variables – we proceed to substitute $\mu+\sigma\,\tilde{x}$ for $\tilde{X}$ and $\bar{r}_m + s_m\tilde{w}$ for $\tilde{r}_m$, where $\tilde{x}$ and $\tilde{w}$ have a bivariate standard normal distribution with correlation ρ. This yields:

$$Q' = s_m\left(\mu - C\right)cov\left(\tilde{I}\,,\tilde{w}\right) + \sigma\, s_m\, cov\left(\tilde{x}\,\tilde{I}\,,\tilde{w}\right)$$

Rewriting the covariance in terms of expectations the term Q' can be further transformed into:

$$Q' = s_m\left(\mu - C\right)E\left(\tilde{I}\,\tilde{w}\right) + \sigma\, s_m E\left(\tilde{x}\, I\,\tilde{w}\right) \tag{A1.1}$$

From Muthén [1990] we know that:

$$\begin{cases} E\left(\tilde{I}\,\tilde{w}\right) = \rho\phi\left(\delta\right) & \text{(A1.2)} \\ E\left(\tilde{x}\,\tilde{I}\,\tilde{w}\right) = \rho\left(\pi\left(\delta\right) + \delta\phi\left(\delta\right)\right) & \text{(A1.3)} \end{cases}$$

Substituting (A1.2) and (A1.3) into (A1.1) we obtain:

$$Q' = \rho\sigma s_m \pi\left(\delta\right) + \rho\left(\mu + \delta\sigma - C\right)s_m\,\phi\left(\delta\right) = \rho\sigma s_m \pi\left(\delta\right)$$

which leads to the desired result:

$$\boxed{\beta^P = \frac{\rho\sigma\pi\left(\delta\right)}{V_e\, s_m}}$$

Appendix 2: The expected value of r_e conditional on r_m

1) $E(r_e \mid \hat{r}_m) = E(\tilde{D}/V_e - 1 \mid \hat{r}_m) = (1+\bar{r}_e)\ \dfrac{E(\tilde{D} \mid \hat{r}_m)}{E(\tilde{D})} - 1$ (A2.1)

where $\bar{r}_e = r_f + \beta^P(\bar{r}_m - r_f)$, and I have used expression (7) from Section II.

2) The problem is reduced to finding an expression for the ratio of conditional and unconditional expectations of liquidating dividends. Since we already have a solution for the denominator in expression (6), we will concentrate now on $E(\tilde{D} \mid \hat{r}_m)$, proceeding as follows:

$$\begin{aligned} E(\tilde{D} \mid \hat{r}_m) &= E(Max(\tilde{X}-C,0) \mid \hat{r}_m) = E(\tilde{X}\tilde{I} \mid \hat{r}_m) - C\,E(\tilde{I} \mid \hat{r}_m) = E(\tilde{X}\tilde{I} \mid \hat{r}_m) - C\,Pr\{\tilde{X} > C \mid \hat{r}_m\} \\ &= E(\tilde{X} \mid \hat{r}_m, \tilde{X} > C)Pr\{\tilde{X} > C \mid \hat{r}_m\} - C\,Pr\{\tilde{X} > C \mid \hat{r}_m\} \\ &= Pr\{\tilde{X} > C \mid \hat{r}_m\}\left[E(\tilde{X} \mid \hat{r}_m, \tilde{X} > C) - C\right] \end{aligned} \quad \text{(A2.2)}$$

where I have used the random variable $\tilde{I}$ that takes the value 1 if $\tilde{X} > C$ and the value 0 if $\tilde{X} \le C$. Here the problem splits in two: (i) the probability of not failing conditional on $\hat{r}_m$; and (ii) expected cash flows conditional on $\hat{r}_m$ and on not failing. In the next two steps we deal consecutively with these two parts of the problem.

3) $Pr\{\tilde{X} > C \mid \hat{r}_m\} = Pr\left\{\dfrac{\tilde{X}-\mu}{\sigma} > \delta \mid \hat{r}_m\right\} = Pr\{\tilde{x} > \delta \mid \hat{r}_m\}$

As in Appendix 1, I have substituted $\mu + \sigma\,\tilde{x}$ for $\tilde{X}$, where $\tilde{x} \approx N(0,1)$. Since $\tilde{x}$ and $\tilde{r}$ behave according to a bivariate normal distribution, the conditional distribution $f(\tilde{x} \mid \hat{r}_m)$ is $N(v,g)$, where $v = \rho(\hat{r}_m - \bar{r}_m)/s_m$ and $g = \sqrt{1-\rho^2}$.[27] Therefore, letting $z = (\delta - v)/g$:

$$Pr\{\tilde{X} > C \mid \hat{r}_m\} = \pi\left(\frac{\delta - v}{g}\right) = \pi(z) \quad \text{(A2.3)}$$

We notice that z is associated with the probability of failing given $\hat{r}_m$ just as δ is associated with the unconditional probability of failing.

4) $E(\tilde{X} \mid \hat{r}_m, \tilde{X} > C) = \mu + \sigma\,E(\tilde{x} \mid \hat{r}_m, \tilde{x} > \delta) = C - \sigma\,\delta + \sigma\,E(\tilde{x} \mid \hat{r}_m, \tilde{x} > \delta)$ (A2.4)

Using again the fact that $f(\tilde{x} \mid \hat{r}_m)$ is $N(v,g)$, and letting $v + g\,\tilde{y} = \tilde{x}$, where $\tilde{y} \approx N(0,1)$, and $z = (\delta - v)/g$, we can rewrite the expectation above as:

$$E(\tilde{x} \mid \hat{r}_m, \tilde{x} > \delta) \quad = \quad E(v + g\,\tilde{y} \mid v + g\,\tilde{y} > \delta) = v + g\; E(\tilde{y} \mid \tilde{y} > z)$$

We have suceeded in transforming the original expectation into the expectation of a standard normal truncated below at z. Using a result available in Maddala [1983][28] we get:

$$E(\tilde{x} \mid \hat{r}_m, \tilde{x} > \delta) \quad = \quad v + g\,\frac{\phi(z)}{\pi(z)} \tag{A2.5}$$

Finally, we substitute (A2.5) back into (A2.4) to obtain:

$$E(\tilde{X} \mid \hat{r}_m, \tilde{X} > C) \quad = \quad C - \sigma\,\delta + \sigma \left\{ v + g\,\frac{\phi(z)}{\pi(z)} \right\} \tag{A2.6}$$

5) In the final two steps we just need to perform substitutions. First (A2.3) and (A2.6) go into (A2.2). Using the definition of $h(\cdot)$ from Section II produces:

$$E(\tilde{D} \mid \hat{r}_m) = -\sigma\; g\; h(z)$$

Relying on (6) for $E(\tilde{D})$ we obtain the ratio of conditional to unconditional expected dividends, denoted k:

$$k = \frac{E(\tilde{D} \mid \hat{r}_m)}{E(\tilde{D})} = (1 - \rho^2)\,\frac{h(z)}{h(\delta)} \tag{A2.7}$$

6) Now (A2.7) goes into (A2.1), and we conclude that:

$$\boxed{E(r_e \mid \hat{r}_m) = -1 + k\,(1 + r_f) + k\;\beta^P\,(\bar{r}_m - r_f)}$$

Appendix 3: An expression for OLS beta

We begin with expression (2), modified by the no-default condition, and use the random variable $\tilde{I}$ that takes the value 1 if $\tilde{X} > C$ and the value 0 if $\tilde{X} \le C$:

$$\beta^Z = \frac{cov\left(Max\left(\tilde{X} - C, 0\right), \tilde{r}_m | \tilde{X} > C\right)}{V_e \, var\left(\tilde{r}_m \mid \tilde{X} > C\right)} = \frac{cov\left(\tilde{X}, \tilde{r}_m \mid \tilde{I} = 1\right)}{V_e \, var\left(\tilde{r}_m \mid \tilde{I} = 1\right)} = \frac{Q''}{V_e \, var\left(\tilde{r}_m \mid \tilde{I} = 1\right)}$$

The term Q'' can be expanded as follows:

$$Q'' = \sigma \, s_m \left\{E\left(\tilde{x}\tilde{w}|\tilde{I} = 1\right) - E\left(\tilde{x} \mid \tilde{I} = 1\right) E\left(\tilde{w}|\tilde{I} = 1\right)\right\}$$

From Bayes' theorem and from Muthén [1990] we obtain:

$$\begin{cases} E\left(\tilde{x} \mid \tilde{I} = 1\right) = \phi(\delta)\pi(\delta)^{-1} & \text{(A3.1)} \\ E\left(\tilde{w}|\tilde{I} = 1\right) = \rho\,\phi(\delta)\pi(\delta)^{-1} & \text{(A3.2)} \\ E\left(\tilde{x}\,\tilde{w}|\tilde{I} = 1\right) = \rho\left[\pi(\delta) + \delta\,\phi(\delta)\right]\pi(\delta)^{-1} & \text{(A3.3)} \\ E\left(\tilde{w}^2 \mid \tilde{I} = 1\right) = \left[\pi(\delta) + \rho^2\delta\,\phi(\delta)\right]\pi(\delta)^{-1} & \text{(A3.4)} \end{cases}$$

With (A3.1) through (A3.4), Q'' and the variance in the denominator become respectively

$$Q'' = \rho\,\sigma\,s_m\,\frac{\pi(\delta)^2 + \phi(\delta)h(\delta)}{\pi(\delta)^2} \qquad \text{(A3.5)}$$

and

$$var\left(\tilde{r}_m \mid \tilde{I} = 1\right) = s_m^2\left\{E\left(\tilde{w}^2 \mid \tilde{I} = 1\right) - E\left(\tilde{w}|\tilde{I} = 1\right)^2\right\} = s_m^2\,\frac{\pi(\delta)^2 + \rho^2\,\phi(\delta)h(\delta)}{\pi(\delta)^2} \qquad \text{(A3.6)}$$

Substituting (A3.5) and (A3.6) into the original expression for β_e^{OLS} we obtain the desired result:

$$\boxed{\beta^Z = \frac{\rho\,\sigma}{V_e\,s_m}\,\frac{\pi(\delta)^2 + \phi(\delta)h(\delta)}{\pi(\delta)^2 + \rho^2\phi(\delta)h(\delta)}}$$

Appendix 4: Comparative statics for equity beta (β^P)

1) In order to sign the partial derivatives of β^P it is useful to begin by establishing the signs of the following terms:

- $h(\delta) < 0$ (A4.1)

  From expression (6) in the text, $h(\delta)$ has opposite the sign of expected dividends. Because of limited liability expected dividends are positive.

- $H(\rho,\delta) < 0$ (A4.2)

  This follows from expression (9) to insure positive stock prices.

- $\pi(\delta)^2 + \rho^2\phi(\delta)h(\delta) > 0$ (A4.3)

  Observe that this term appears in Appendix 3, expression (A3.6) for $var(\tilde{r}_m \mid \tilde{I} = 1)$. The result is a consequence of the positive sign of the variance.

- $\pi(\delta)^2 + \phi(\delta)h(\delta) > 0$ (A4.4)

  This is because (A4.3) holds for any value of ρ and δ is independent of ρ. Just set ρ=1 in (A4.3).

2) Now we are ready to sign the partial derivatives of β^P with respect to δ, C, σ and r_f .

- $$\frac{\partial \beta^P}{\partial \delta} = \frac{\rho(1+r_f)\left[\pi(\delta)^2 + \phi(\delta)h(\delta)\right]}{s_m H_m(\rho,\delta)^2}$$

  Because of (A4.4) the partial derivative of β^P with respect to δ has the sign of the correlation coefficient ρ. Since δ is strictly increasing in probability of bankruptcy, the partial of β^P with respect to p also has the sign of ρ. The graph of systematic risk of equity versus p and the ρ in Figure 2 shows that the absolute value of the first derivative of beta with respect to p at first decreases with p, reaches a minimum at, say, p^*, and then increases with p. This indicates that: (i) for $\rho > 0$ the second derivative with respect to p is positive beyond p^*; (ii) $\partial^2 \beta_e / \partial p^2 > 0$; and (iii) beta is likely to become less stable as p increases.

■ $\dfrac{\partial \beta^P}{\partial C}$

From the previous paragraph we conclude that: (i) the partial with respect to leverage C has the sign of ρ (because $\partial\delta/\partial C = 1$); (ii) that $\partial^2 \beta^P / \partial C^2 > 0$; and (iii) that beta is likely to become less stable as C increases. The last result is in agreement with DeJong and Collins [1985].

■ $$\frac{\partial \beta^P}{\partial \sigma} = \frac{-\rho(C-\mu)(1+r_f)\left[\pi(\delta)^2 + \phi(\delta)h(\delta)\right]}{\sigma^2 s_m H_m(\rho,\delta)^2}$$

Because of (A4.4) the partial with respect to business risk σ has the sign of the correlation coefficient ρ if $C < \mu$; has opposite the sign of the correlation coefficient if $C > \mu$; and equals zero when $C = \mu$. Thus, for relatively *low levels of debt and likelihood of bankruptcy*, beta is increasing (decreasing) in business risk when the correlation coefficient is positive (negative). However, for relatively *high levels of debt and likelihood of bankruptcy*, beta is decreasing (increasing) in business risk when the correlation coefficient is positive (negative). The existence of a positive association between specific risk and beta has been confirmed in the literature. Beaver, Kettler, and Scholes [1970] observed a strong positive linkage between earnings variability and beta. Hawawini and Keim [1992] provide evidence consistent with this result for NYSE and AMEX stocks, for the period April 1951 through December 1989. The present model raises the possibility of a negative association between specific risk and beta, and therefore implies the need to control for correlation with the market in tests of the association between beta and business risk.

■ $$\frac{\partial \beta^P}{\partial r_f} = \frac{-\rho\pi(\delta)\left[h(\delta) + \rho\pi(\delta)\dfrac{(1+\bar{r}_m)}{s_m}\right]}{s_m H_m(\rho,\delta)^2}$$

Case $\rho < 0$: the full term in brackets must be negative because of (A4.1). This means that the partial with respect to the risk-free rate is always negative in this case.

Case $\rho > 0$: the partial with respect to the risk-free rate will be negative so long as

$$\frac{\rho\,\pi(\delta)}{|h(\delta)|} > \frac{s_m}{1+\bar{r}_m}$$; otherwise, it will be positive.

- $$\frac{\partial \beta^P}{\partial \rho} = \frac{-\pi(\delta)h(\delta)(1+r_f)}{s_m H_m(\rho,\delta)^2}$$

Because of (A4.1) the partial with respect to the correlation coefficient ρ is always positive.

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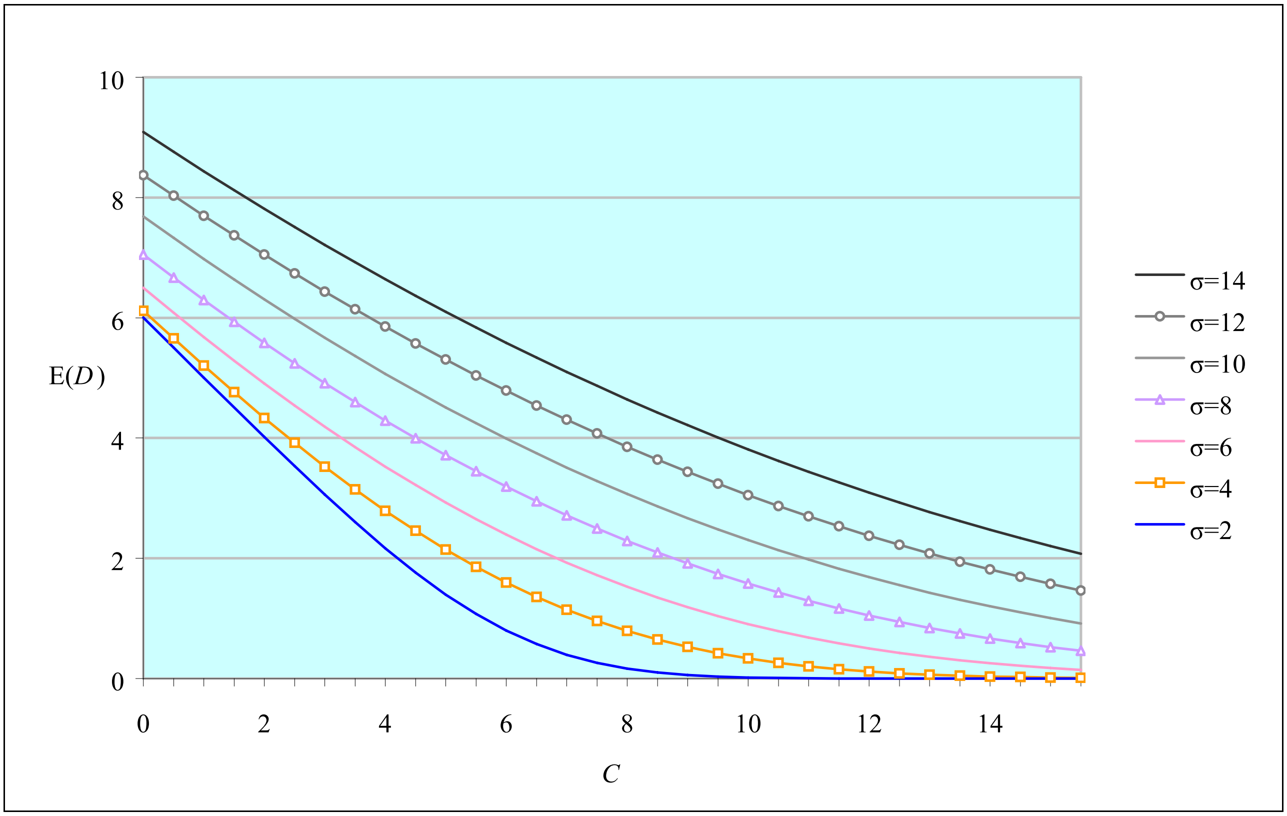

Figure 1: Expected dividends as a function of debt (C) and business risk (σ). The functional relationship is expression (6) in the text. The graph was drawn assuming the expected value of gross cash-flows to be μ = 6. Lines are drawn from top to bottom in the order shown in the legend. • Expected dividends increase with business risk given the level of debt, and decrease with debt given the level of business risk. The behavior of expected dividends with respect to business risk is due to the fact that under limited liability shareholders exercise their option to abandon assets to creditors whenever cash-flows are insufficient to meet debt payments.

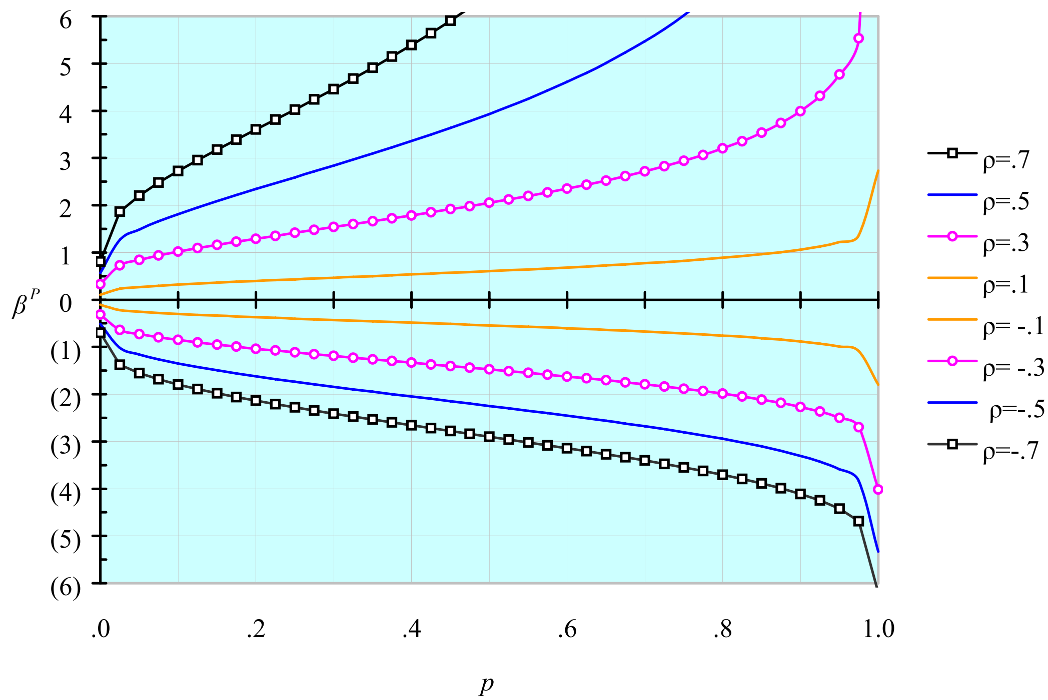

Figure 2: Bankruptcy-adjusted beta (β^P) versus probability of bankruptcy (p) and correlation with the market (ρ). The functional relationship for β^P is expression (8) in the text. The graph was drawn assuming that $r_m = .15$, $s_m = .23$, and $r_f = .05$. Lines are drawn from top to bottom in the order shown in the legend; lines above the horizontal axis are for positive ρ and those below are for negative ρ. • We observe in the graph that: (i) β^P increases (decreases) with p when ρ is positive (negative); and (ii) that β^P increases with ρ at any given level of p. These results are demonstrated analitically in Appendix 4. Although not apparent from the graph, β^P is not defined when p=0 or p=1. For any other probability of bankruptcy $\rho = 0 \Rightarrow \beta^P = 0$.

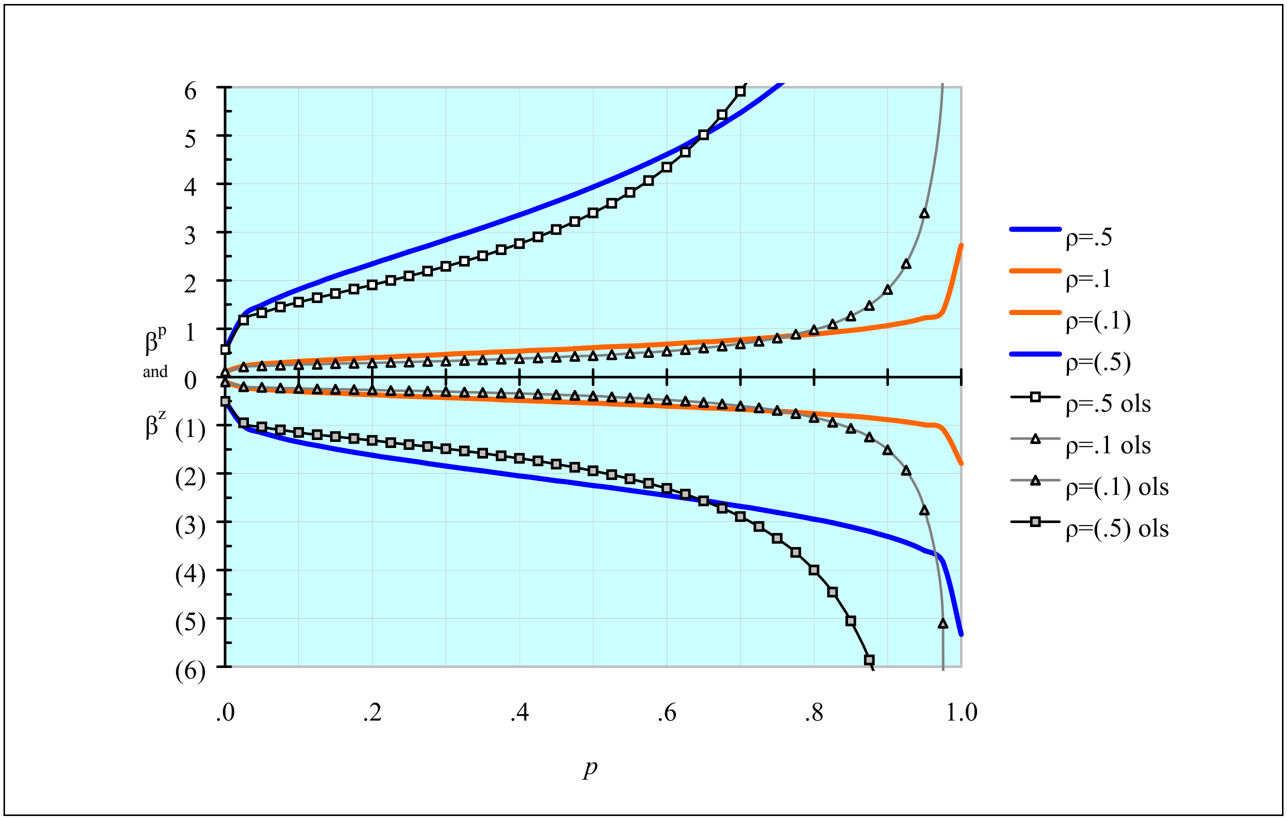

Figure 3: OLS (β^{Z}) and bankruptcy-adjusted betas (β^{P}) versus probability of bankruptcy (p) and correlation with the market (ρ). The functional relationships are expression (8) for bankruptcy-adjusted beta, and expression (14) for OLS beta. The graph was drawn assuming that $r_m = .15$, $s_m = .23$, and $r_f = .05$. Lines are drawn from top to bottom in the order shown in the legend, thick lines for β^{P} and thin lines with a symbol for β^{Z}; lines above the horizontal axis are for positive ρ and those below for negative ρ. • This graph shows that β^{Z} can overestimate or underestimate β^{P}, depending on the values of ρ and p. The lines corresponding to β^{P} are a replica of those in Figure 2, for the given levels of ρ. Although not apparent from the graph, β^{Z} is not defined when p=0 or p=1. For any other probability of bankruptcy $\rho = 0 \Rightarrow \beta^{Z} = 0$.

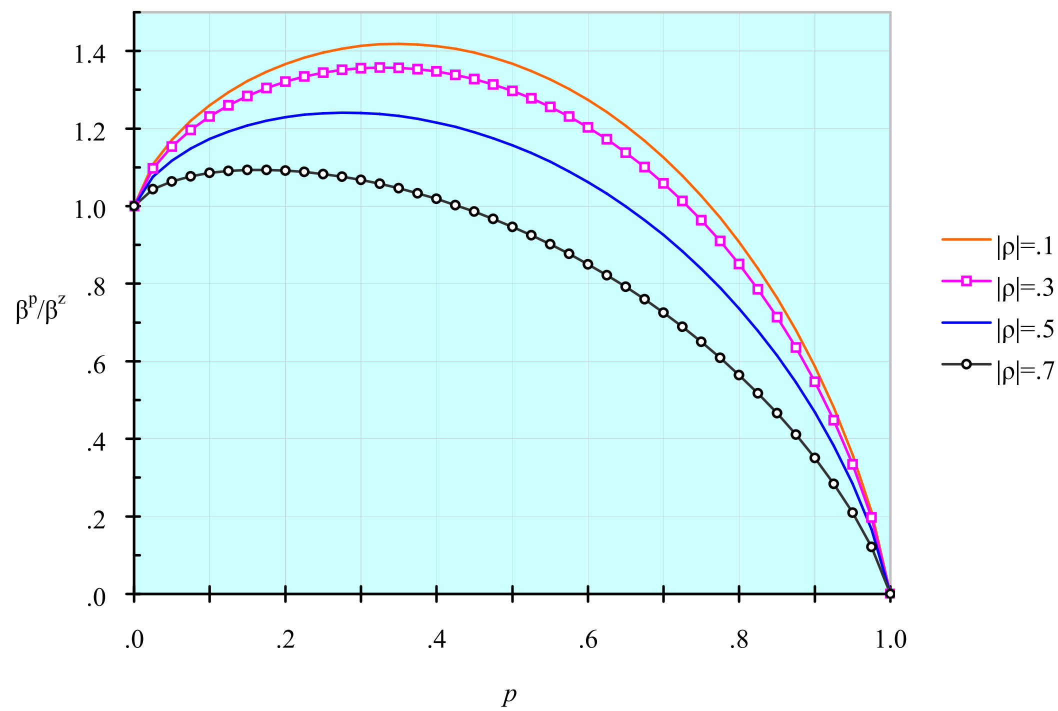

Figure 4: The ratio of bankruptcy-adjusted beta (β^P) to OLS beta (β^Z) versus probability of bankruptcy (p) and correlation with the market (ρ). The functional relationship is expression (15) in the text. Lines are drawn from top to bottom in the order shown in the legend. Lines for negative and positive correlations with the same absolute value coincide and the line for $\rho = 0$ is not drawn because it virtually coincides with that for $\rho = 0.1$. • From the graph we observe that: (i) the ratio β^P / β^Z tends to one as p tends to zero, and to zero as p tends to one; (ii) the worst deficit in β^Z happens when $\rho = 0$ and $p \cong .35$; (iii) given ρ there is a critical p^* such that $\beta^Z < \beta^P$ for all $p \in (0, p^*)$, $\beta^Z = \beta^P$ for $p = p^*$; and $\beta^Z > \beta^P$ for all $p \in (p^*, 1)$; (iv) given p, the ratio β^P / β^Z is a decreasing function of ρ. It is interesting that, although β^P and β^Z depend on the market parameters r_f, s_m and $\bar{r}_m$, the ratio between them does not.

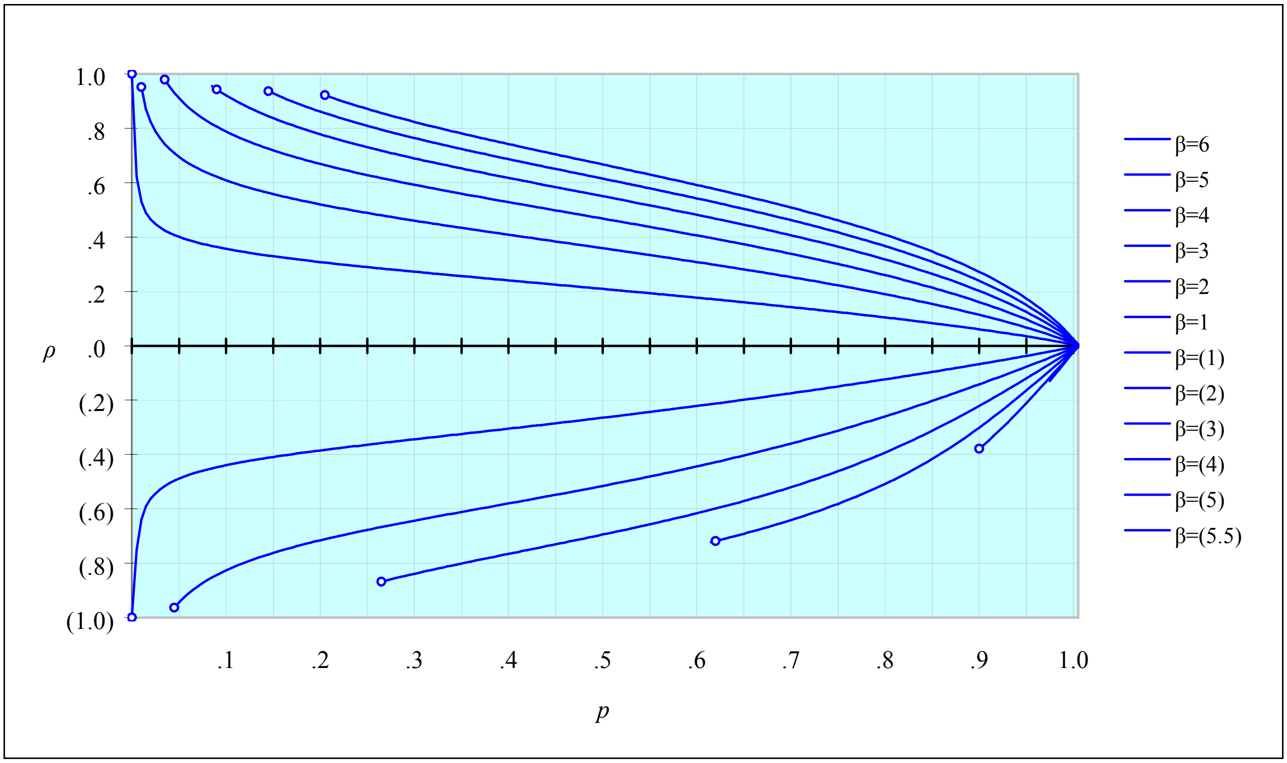

Figure 5: Correlation with the market (ρ) versus OLS beta (β^z) and probability of bankruptcy (p). The graph shows ρ as a function of p and β^z, with ρ being obtained by means of numerical methods from the expression for β^z (expression 14 in the text). Lines are drawn from top to bottom, for each value of β^z and in the order shown in the legend. • We observe that: (i) ρ tends to 0 as $p \to 1$, and it does so faster the closer we are to p=1 and the higher the absolute value of β^z; (ii) ρ tends to 1 as $p \to 0$, and it does so faster the closer we are to p=0 and the lower the absolute value of β^z; (iii) there are combinations of p and β^z values for which ρ does not exist.

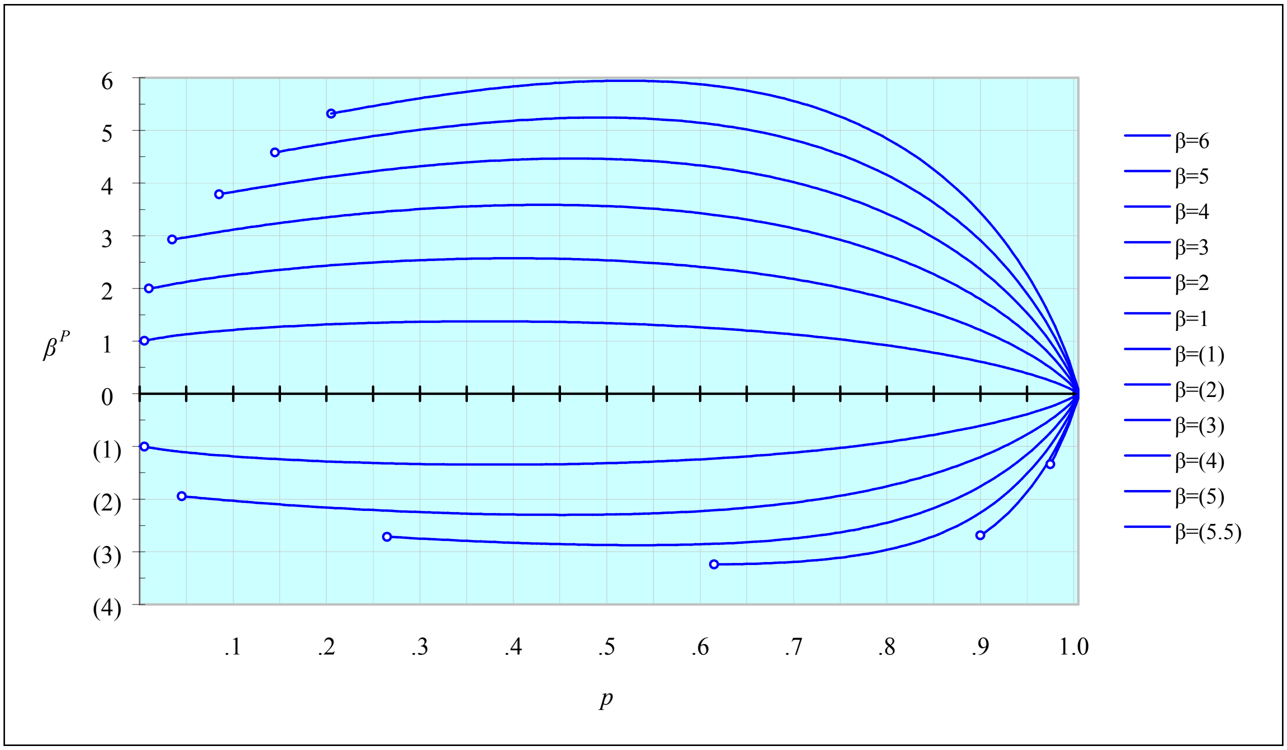

Figure 6: Bankruptcy-adjusted beta (β^P) versus OLS beta (β^Z) and probability of bankruptcy (p). This graph shows β^P as a function of p and β^Z, with ρ being obtained from the expression for β^Z (number 14 in the text) by means of numerical methods. Lines are drawn for from top to bottom, for each value of OLS beta and in the order shown in the legend. • We observe that: (i) β^P collapses towards zero as $p \to 1$; (ii) there are combinations of p and β^Z values for which β^P does not exist; (iii) the range of probabilities over which β^Z underestimates β^P tends to shrink as the absolute value of β^Z increases, and there is a value of β^Z above which OLS always overestimates β^P.

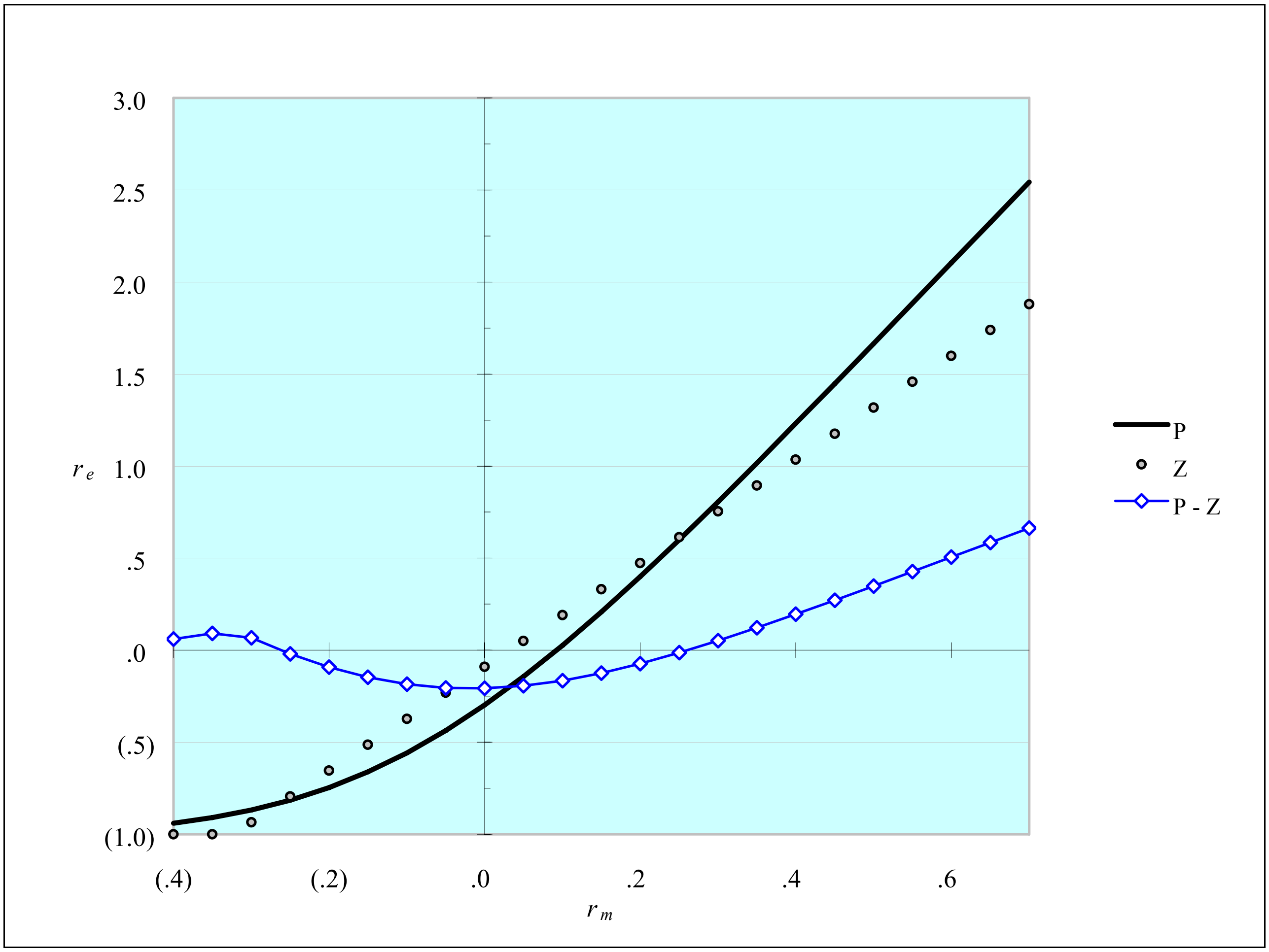

Figure 7: Expected returns conditional on the market's return (r_m) admitting (model *P*) or ignoring (model *Z*) the possibility of bankruptcy - Example # 1. Admitting bankruptcy expected returns are given by $E(\tilde{r}_e \mid \hat{r}_m) = -1 + k\,(1 + r_f) + k\,\beta^P\,(\bar{r}_m - r_f)$. Ignoring bankruptcy by $E(\tilde{r}_e \mid \hat{r}_m) = r_f + \beta^Z(\hat{r}_m - r_f)$. This example was solved with p = 25.2%, ρ = 60%, $r_m = .15$, $s_m = .23$ and $r_f = .05$. Systematic risk values according to the model are: β^P = 3.29 and β^Z = 2.82, their ratio being $\Xi = 1.168$. • The graph shows that conditional expected returns according to model *P* are less than those under the traditional view for $\hat{r}_m \in (-.25, +.25)$, being higher than those under the traditional view outside that range. That is, an analyst computing expected returns according to model *Z* would sometimes believe that actual returns are too high, sometimes too low depending on the value of $\hat{r}_m$, if returns are actually generated by model *P*. The value p = 25.2% is at the high end of probabilities of failure during the period 1973-1980 when returns for the most distressed firms were not found to be less than average market returns. • Notice that the graph of model *P*'s conditional expected returns approaches the –1 boundary asymptotically, but model Z's lower bound is imposed exogenously.

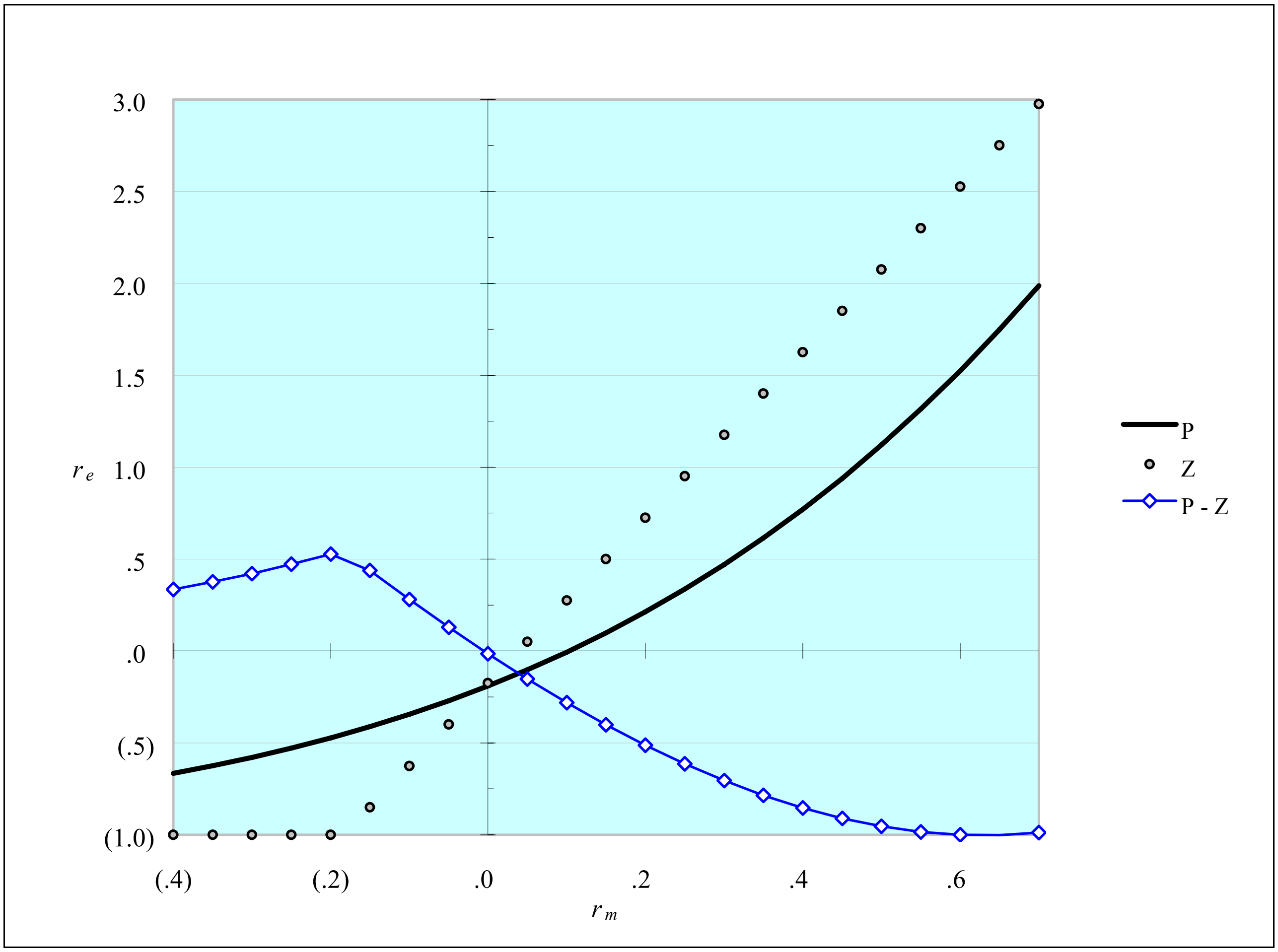

Figure 8: Expected returns conditional on the market's return (r_m) admitting (model *P*) or ignoring (model *Z*) the possibility of bankruptcy - Example # 2. Admitting bankruptcy expected returns are given by $E(\tilde{r}_e \mid \hat{r}_m) = -1 + k\,(1 + r_f) + k\;\beta^P\,(\bar{r}_m - r_f)$. Ignoring bankruptcy by $E(\tilde{r}_e \mid \hat{r}_m) = r_f + \beta^Z(\hat{r}_m - r_f)$. This example was solved with p = 90.9%, ρ = 20%, $r_m = .15$, $s_m = .23$ and $r_f = .05$. Systematic risk values according to the model are: β^P = 2.42 and β^Z = 4.50, their ratio being $\Xi = .537$. • The graph shows that model *P* conditional expected returns exceed those under the traditional view for all $\hat{r}_m$ over about zero. That is, an analyst calculating expected returns with model *Z* would believe that actual returns are too low when $\hat{r}_m > 0$, and too high when $\hat{r}_m < 0$, if returns are in fact generated by model *P*. The value p = 90.9% is at the high end of probabilities of failure during the period 1981-1998 when returns for the most distressed firms were found to be less than average market returns. Thus the graph is consistent with bankruptcy's risk-return paradox. • Notice that the graph of model *P*'s conditional expected returns approaches the –1 boundary asymptotically, but model Z's lower bound is imposed exogenously.

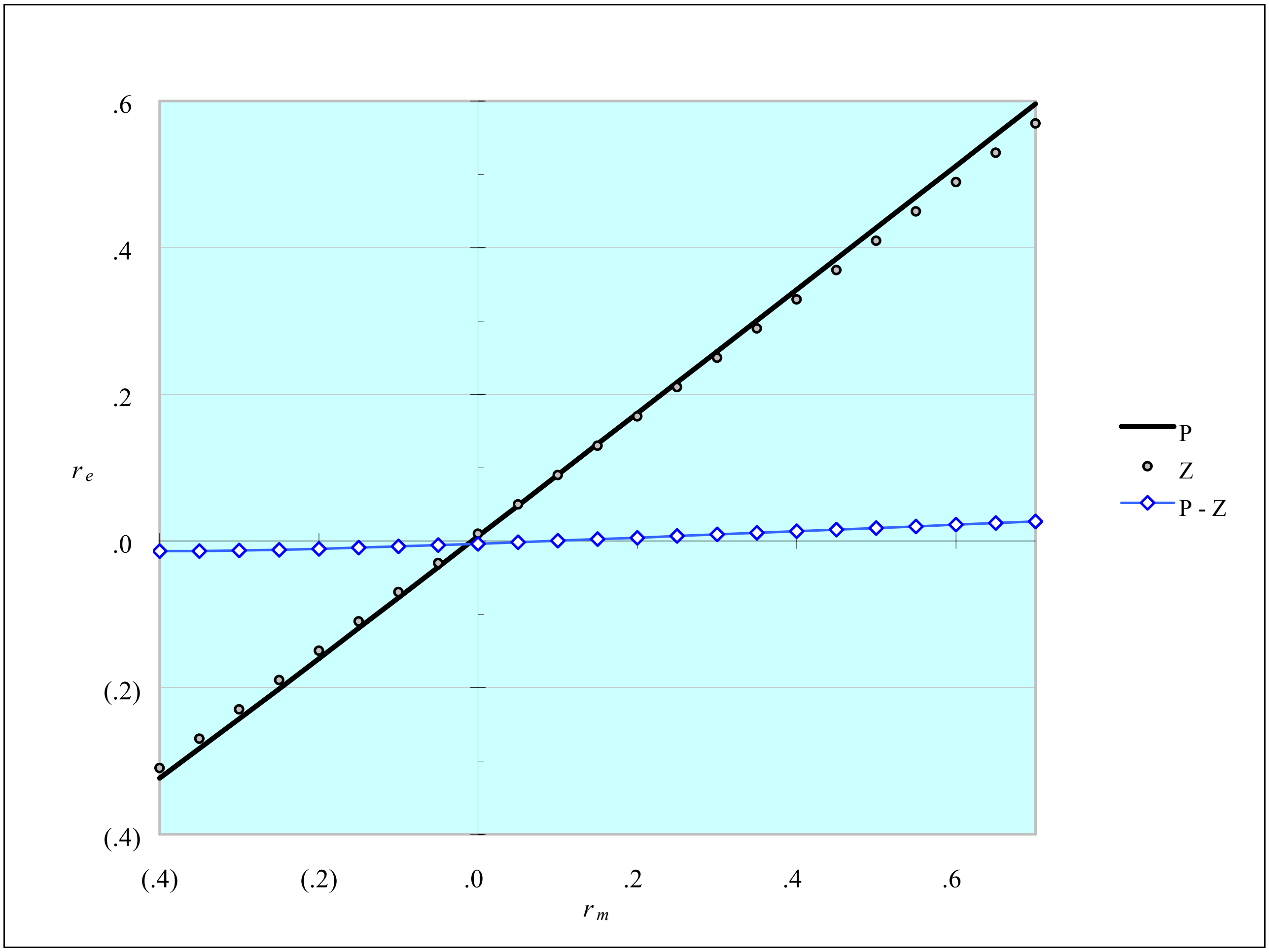

Figure 9: Expected returns conditional on the market's return (r_m) admitting (model *P*) or ignoring (model *Z*) the possibility of bankruptcy - Example # 3. Admitting bankruptcy expected returns are given by $E(\tilde{r}_e \mid \hat{r}_m) = -1 + k\,(1 + r_f) + k\,\beta^P\,(\bar{r}_m - r_f)$. Ignoring bankruptcy by $E(\tilde{r}_e \mid \hat{r}_m) = r_f + \beta^Z(\hat{r}_m - r_f)$. This example was solved with p = 1%, ρ = 40%, $r_m = .15$, $s_m = .23$ and $r_f = .05$. Systematic risk values according to the model are: β^P = .84 and β^Z = .80, their ratio being $\Xi = 1.046$. • The graph shows that conditional expected returns according to model *P* approximate those under the traditional view, being slightly above when $\hat{r}_m$ is positive. An analyst computing expected returns using model *Z* would not find significant discrepancies if returns are, in fact, generated according to model *P*. • Although behavior of expected returns close to the –1 bankruptcy boundary is not shown in this figure, model *P*'s conditional expected returns always approach that boundary asymptotically, but model *Z*'s never do.

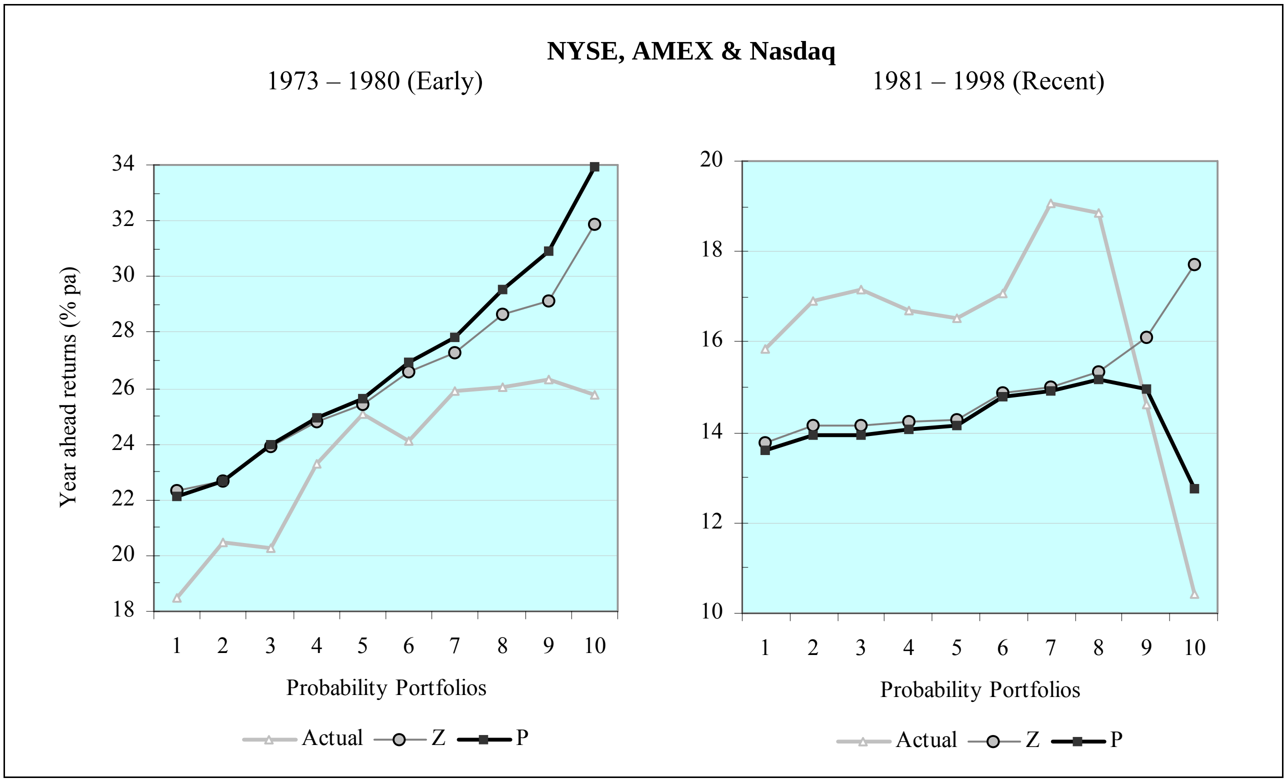

Figure 10: Relationship between probability of bankruptcy, actual returns and returns predicted by models *Z* and *P*. The plot on the left covers the period January 1973 until December 1980, and the plot on the right the period January 1981 until January 1999. Probability of bankruptcy is computed with Ohlson's [1980] model, with the necessary data assumed to be available at fiscal year end. At the beginning of each month firms are ranked according to increasing risk of bankruptcy and assigned to portfolios 1 through 10, so that each portfolio contains one decile of firms by probability. Returns are accumulated by compoundig over the twelve month period following portfolio formation. Actual returns include delisting returns, with missing delisting returns being replaced by –30% and -50% respectively for NYSE-AMEX and Nasdaq firms. Expected returns under model *Z* (zero probability of bankruptcy) are computed with the traditional CAPM formula using OLS beta. Expected returns under model *P* (positive probability of bankruptcy) are computed with expression (11) in the text. The data used to plot these graphs is given in Table III.

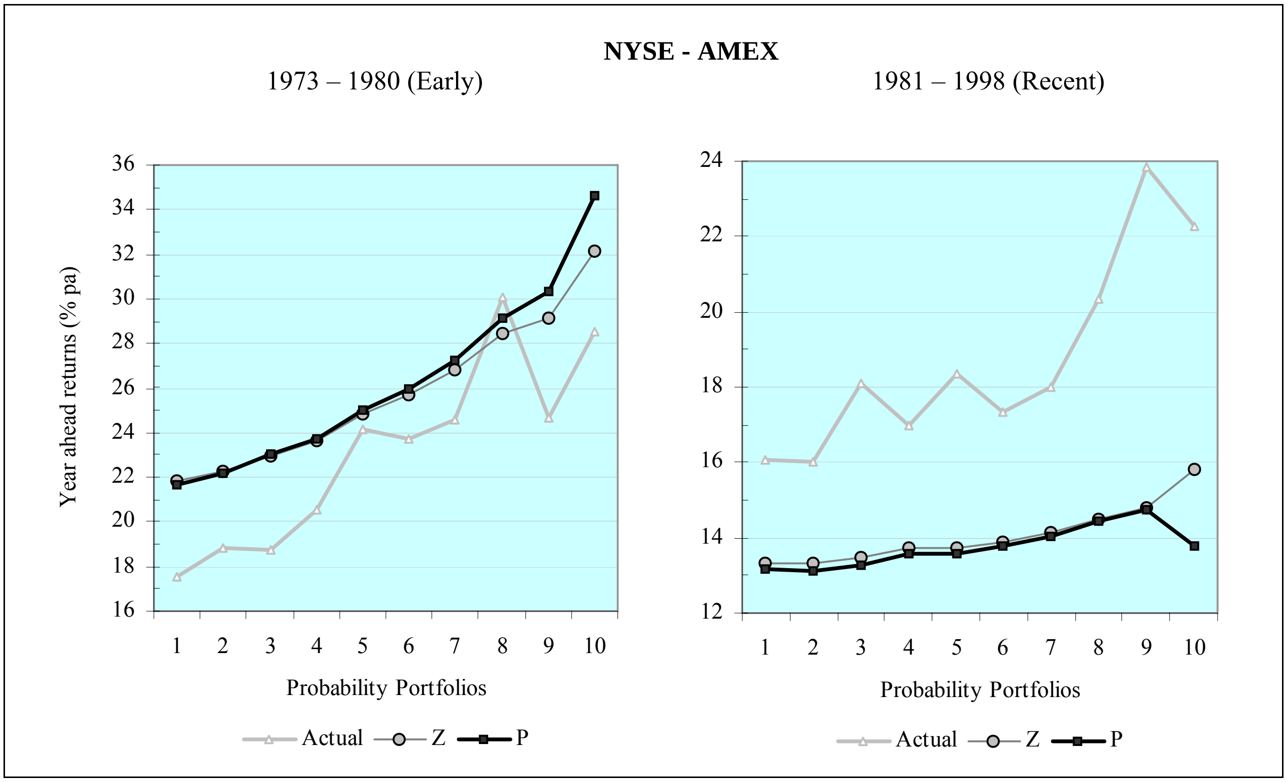

Figure 11: Relationship between probability of bankruptcy, actual returns and returns predicted by models *Z* and *P*. The plot on the left covers the period January 1973 until December 1980, and the plot on the right the period January 1981 until January 1999. Probability of bankruptcy is computed with Ohlson's [1980] model, with the necessary data assumed to be available at fiscal year end. At the beginning of each month firms are ranked according to increasing risk of bankruptcy and assigned to portfolios 1 through 10, so that each portfolio contains one decile of firms by probability. Returns are accumulated by compoundig over the twelve month period following portfolio formation. Actual returns include delisting returns, with missing delisting returns being replaced by –30% for firms listed by the NYSE or AMEX. Expected returns under model *Z* (zero probability of bankruptcy) are computed with the traditional CAPM formula using OLS beta. Expected returns under model *P* (positive probability of bankruptcy) are computed with expression (11) in the text. The data used to plot these graphs is given in Table IV.

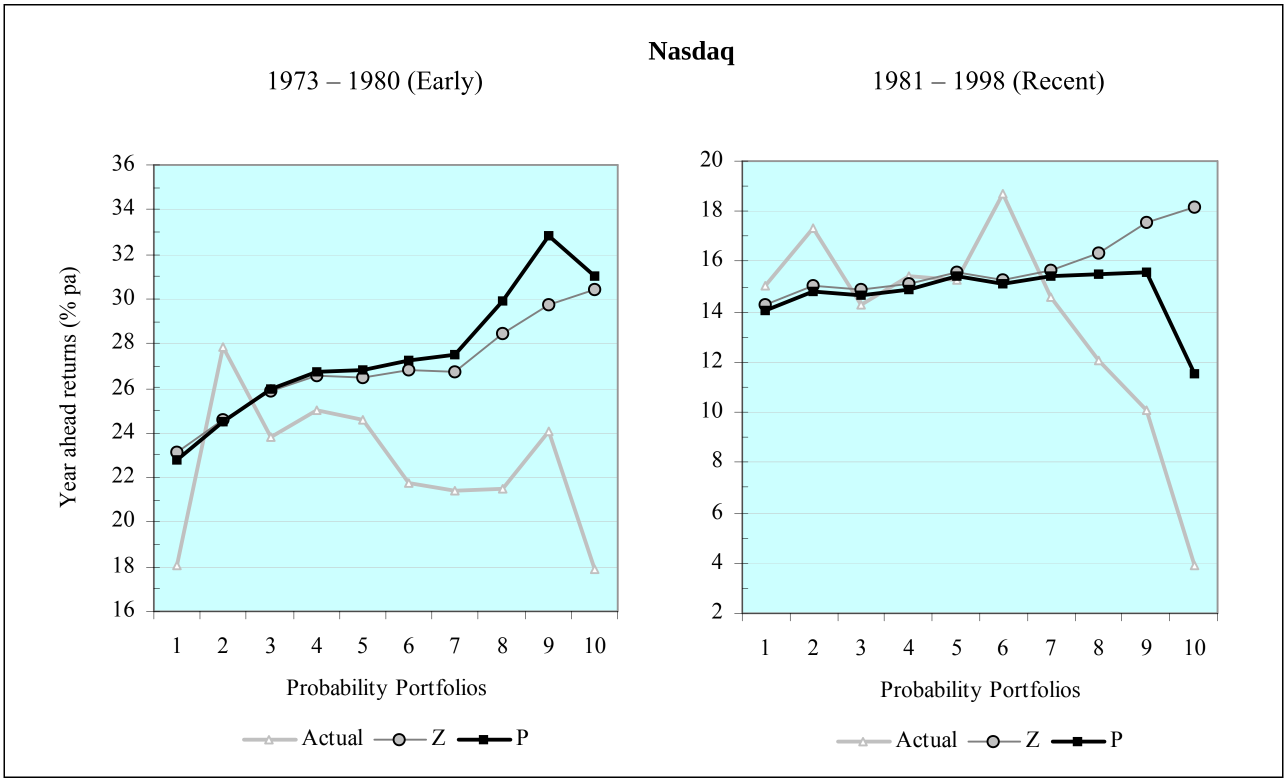

Figure 12: Relationship between probability of bankruptcy, actual returns and returns predicted by models *Z* and *P*. The plot on the left covers the period January 1973 until December 1980, and the plot on the right the period January 1981 until January 1999. Probability of bankruptcy is computed with Ohlson's [1980] model, with the necessary data assumed to be available at fiscal year end. At the beginning of each month firms are ranked according to increasing risk of bankruptcy and assigned to portfolios 1 through 10, so that each portfolio contains one decile of firms by probability. Returns are accumulated by compoundig over the twelve month period following portfolio formation. Actual returns include delisting returns, with missing delisting returns being replaced by -50% for firms listed by the Nasdaq. Expected returns under model *Z* (zero probability of bankruptcy) are computed with the traditional CAPM formula using OLS beta. Expected returns under model *P* (positive probability of bankruptcy) are computed with expression (11) in the text. The data used to plot these graphs is given in Table V.

Table I

Descriptive Statistics for Test Variables by Time Period and Stock Exchange Listing

Means and standard deviations for test variables are provided according to stock exchange listing and by time period. • *N* is the mean number of firms in the sample per month; probability of bankruptcy is computed with Ohlson's [1980] model, with the necessary data assumed to be available at fiscal year end; *BME* is the ratio of book equity to market equity, both publicly available at the time (it is assumed that information becomes available six months after fiscal year end); Size is log of market equity at the end of the most recent known fiscal year in millions of dollars; returns are monthly returns from CRSP including delisting returns, with missing delisting returns replaced by –30% and –50% respectively for NYSE-AMEX and Nasdaq firms.

Time Periods		N	Size		BME		Pr{failure} (%)		Returns (% pm)	
			Mean	StDev	Mean	StDev	Mean	StDev	Mean	StDev
Panel A: *NYSE-AMEX* & *Nasdaq*										
70's	(1973-1980)	2,476	3.159	1.821	1.339	.988	5.7	17.0	1.74	16.11
80's	(1981-1990)	2,730	3.627	1.879	.848	.669	13.8	27.7	.70	17.31
90's	(1991-1998)	3,327	4.082	1.989	.737	.651	16.4	30.0	1.61	19.54
80's + 90's		2,997	3.866	1.951	.790	.662	15.1	28.9	1.15	18.45
All		2,837	3.722	1.946	.908	.778	12.5	26.5	1.31	17.86
Panel B: *NYSE-AMEX*										
70's	(1973-1980)	1,514	3.675	1.839	1.313	.950	3.4	12.3	1.61	14.42
80's	(1981-1990)	1,230	4.482	1.884	.917	.652	5.8	17.0	1.32	13.19
90's	(1991-1998)	1,347	4.949	2.045	.775	.641	7.6	19.3	1.67	13.38
80's + 90's		1,282	4.715	1.980	.847	.650	6.6	18.1	1.48	13.28
All		1,353	4.442	1.997	.976	.774	5.5	16.4	1.53	13.69
Panel C: *Nasdaq*										
70's	(1973-1980)	962	2.164	1.297	1.388	1.057	9.4	21.9	1.94	18.46
80's	(1981-1990)	1,500	2.768	1.428	.777	.679	20.5	32.7	.19	20.04
90's	(1991-1998)	1,980	3.377	1.633	.706	.658	21.8	34.0	1.57	22.80
80's + 90's		1,715	3.104	1.574	.738	.668	21.2	33.4	.90	21.52
All		1,484	2.970	1.572	.836	.776	18.8	31.8	1.11	20.95

Table II

Descriptive Statistics for Traditional and Bankruptcy-Adjusted Betas, for the Coefficient of Correlation with the Market and for OLS Beta F-tests by Time Period and Stock Exchange Listing

Means and standard deviations for the indicated variables are provided according to stock exchange listing and by time period. • *N* is the mean number of firms in the sample per month; OLS beta is based on the 60 preceeding months (at least 24 required) using one Dimson lag [1979] to correct for infrequent trading, and using Shumway [1997] and Shumway and Warther's [1999] corrections for missing delisting returns; model *P* betas (β^p) are estimated from OLS betas (β^z) using the procedure described in Section III; ρ *is* the correlation between cash flows and the market, inferred from β^z using the procedure described in Section III; the last two columns contain means and standard deviations of the p-values associated with F-tests performed for the regressions used to estimate β^z.

Time Periods		N	OLS beta (β^z)		Model *P* (β^p)		ρ (%)		OLS beta F-test (%)	
			Mean	StDev	Mean	StDev	Mean	StDev	Mean	StDev
Panel A: *NYSE-AMEX* & *Nasdaq*										
70's	(1973-1980)	2,301	1.005	.513	1.017	.512	44.5	19.4	3.82	12.52
80's	(1981-1990)	2,811	1.011	.629	1.019	.588	39.1	21.3	8.20	18.32
90's	(1991-1998)	3,236	1.003	.980	.916	.776	33.8	26.2	16.30	24.37
80's + 90's		3,001	1.007	.817	.969	.688	36.5	24.0	12.10	21.83
All		2,786	1.007	.752	.982	.647	38.6	23.1	10.01	20.21
Panel B: *NYSE-AMEX*										
70's	(1973-1980)	1,553	.980	.464	.990	.470	45.1	18.2	2.21	9.26
80's	(1981-1990)	1,343	.904	.451	.938	.446	40.4	17.8	3.72	12.00
90's	(1991-1998)	1,314	.779	.678	.767	.609	32.0	23.1	12.12	21.38
80's + 90's		1,330	.849	.566	.861	.532	36.6	20.8	7.43	17.31
All		1,399	.894	.537	.906	.515	39.6	20.4	5.66	15.26
Panel C: *Nasdaq*										
70's	(1973-1980)	748	1.058	.599	1.079	.592	43.1	21.7	7.16	16.96
80's	(1981-1990)	1,468	1.109	.742	1.102	.694	37.7	24.3	12.30	21.82
90's	(1991-1998)	1,922	1.155	1.116	1.029	.866	35.3	28.3	19.15	25.83
80's + 90's		1,671	1.133	.953	1.064	.789	36.4	26.5	15.82	24.21
All		1,388	1.121	.905	1.067	.758	37.6	25.9	14.39	23.39

Table III

Portfolio Results for the Relation between Probability of Bankruptcy, Size, Book-to-Market, OLS Beta, Bankruptcy-Adjusted Beta, Correlation with the Market and Year-Ahead Returns: Actual and According to Models *Z* and *P*
- NYSE, AMEX and Nasdaq Firms -

Average characteristics are given from January 1973 to December 1980 and from January 1981 to January 1999 for each of ten portfolios and for all portfolios. • Firms are sorted in order of probability of bankruptcy at the beginning of each month and assigned to portfolios 1 through 10 so that each portfolio contains one decile. Portfolio 10 has the most distressed firms. • The main observation is that expected returns according to model *P* (that admits bankruptcy) and model Z (that ignores bankruptcy) diverge at the high end of failure probabilities during the more recent period. • *N* is the average number of firms in the portfolio; *p* is probability of bankruptcy from Ohlson's [1980] model, with the required data assumed to be available at fiscal year end; *BME* is the ratio of book to market equity, both publicly available at the time (it is assumed that information is available six months after fiscal year end); Size is log of market equity at the end of the most recent known fiscal year in millions of dollars; OLS beta (β^Z) is based on the 60 preceeding months (at least 24 required) using one Dimson lag [1979], and using Shumway [1997] and Shumway and Warther's [1999] corrections for missing delisting returns; model *P* betas (β^P) are estimated from β^Z using the procedure described in Section III; model *P conditional* betas are obtained by multiplying β^P by the ratio of conditional to unconditional expected dividends (k); ρ *is* the correlation between cash flows and the market, inferred from β^Z using the procedure described in Section III; actual returns are compound returns accumulated over the twelve months following portfolio formation, including delisting returns, with missing delisting returns replaced by –30% and –50% respectively for NYSE-AMEX and Nasdaq firms; expected returns under model *Z* are twelve month returns computed with the traditional CAPM formula and β^Z; expected returns under model *P* are computed with expression (11).

Port-folio	N	Firm Characteristics			Beta			ρ	Actual Returns	Model Returns	
		p (%)	BME	Size	β^Z	β^P	Condi-tional β^P	(%)		*Z* (% pa)	*P* (% pa)
					Panel A: *1973 until 1980 (Early Period)*						
1	186	.03	.893	4.51	.75	.75	.81	49.6	18.5	22.3	22.1
2	187	.09	1.071	4.54	.77	.78	.84	45.8	20.5	22.7	22.7
3	187	.17	1.171	4.42	.82	.83	.91	45.7	20.3	23.9	23.9
4	187	.27	1.309	4.06	.88	.89	.97	46.4	23.3	24.8	25.0
5	187	.40	1.332	3.73	.93	.94	1.03	46.8	25.1	25.4	25.7
6	187	.60	1.463	3.34	.99	1.01	1.11	47.4	24.1	26.6	27.0
7	187	.90	1.524	2.98	1.04	1.07	1.18	47.2	25.9	27.2	27.8
8	187	1.47	1.519	2.64	1.11	1.16	1.29	47.0	26.0	28.6	29.5
9	187	3.24	1.569	2.15	1.16	1.26	1.40	44.3	26.3	29.1	30.9
10	186	31.41	1.569	1.79	1.27	1.42	1.62	30.3	25.7	31.9	33.9
All	1,868	3.85	1.342	3.42	.97	1.01	1.12	45.0	23.6	26.3	26.8
					Panel B: *1981 until 1998 (Recent Period)*						
1	224	.02	.601	4.96	.78	.79	.77	51.0	15.8	13.8	13.6
2	225	.10	.680	5.29	.82	.82	.81	46.7	16.9	14.1	13.9
3	225	.22	.710	5.31	.82	.83	.81	43.4	17.2	14.1	14.0
4	224	.42	.755	5.05	.85	.87	.85	42.2	16.7	14.2	14.1
5	224	.73	.829	4.64	.88	.90	.88	40.6	16.5	14.3	14.1
6	225	1.30	.867	4.21	.94	.98	.96	40.3	17.1	14.9	14.8
7	225	2.69	.890	3.72	1.00	1.07	1.04	38.8	19.1	15.0	14.9
8	224	7.39	.937	3.30	1.06	1.22	1.16	35.2	18.8	15.3	15.2
9	225	25.96	.864	2.91	1.18	1.49	1.38	29.1	14.6	16.1	14.9
10	224	78.08	.538	2.69	1.35	.90	.85	10.7	10.4	17.7	12.8
All	2,244	11.68	.767	4.21	.97	.99	.95	37.8	16.3	15.0	14.2

Table IV

Portfolio Results for the Relation between Probability of Bankruptcy, Size, Book-to-Market, OLS Beta, Bankruptcy-Adjusted Beta, Correlation with the Market and Year-Ahead Returns: Actual and According to Models *Z* and *P*
- NYSE-AMEX Firms -

Average characteristics are given from January 1973 until December 1980 and from January 1981 until January 1999 for each of ten portfolios and for all portfolios combined. • Firms are sorted in order of probability of bankruptcy at the beginning of each month and assigned to portfolios 1 through 10 so that each portfolio contains one decile. Portfolio 10 contains the most distressed firms. • The main observation is that expected returns according to model *P* (that admits bankruptcy) and model Z (that ignores bankruptcy) diverge at the high end of failure probabilities during the more recent period. • *N* is the average number of firms in the portfolio; *p* is probability of bankruptcy computed with Ohlson's [1980] model, with the necessary data assumed to be available at fiscal year end; *BME* is the ratio of book equity to market equity, both publicly available at the time (it is assumed that information becomes available six months after fiscal year end); Size is log of market equity at the end of the most recent known fiscal year in millions of dollars; OLS beta (β^Z) is based on the 60 preceeding months (at least 24 required) using one Dimson lag [1979] to correct for infrequent trading, and using Shumway [1997] and Shumway and Warther's [1999] corrections for missing delisting returns; model *P* betas (β^P) are estimated from β^Z using the procedure described in Section III; model *P conditional* betas are obtained by multiplying β^P by the ratio of conditional to unconditional expected dividends (*k*); ρ *is* the correlation between cash flows and the market, inferred from β^Z using the procedure described in Section III; actual returns are compound returns accumulated over the twelve month period following portfolio formation, including delisting returns, with missing delisting returns replaced by –30% and –50% respectively for NYSE-AMEX and Nasdaq firms; expected returns under model *Z* are twelve month returns computed with the traditional CAPM formula using β^Z; expected returns under model *P* are computed with expression (11).

Port-folio	N	Firm Characteristics			Beta			ρ	Actual Returns	Model Returns	
		p (%)	BME	Size	β^Z	β^P	Conditional β^P	(%)		*Z* (% pa)	*P* (% pa)
Panel A: *1973 until 1980 (Early Period)*											
1	130	.03	.845	4.93	.73	.73	.79	48.2	17.5	21.8	21.6
2	131	.08	1.009	4.92	.76	.76	.82	45.2	18.9	22.2	22.2
3	131	.15	1.129	4.94	.78	.79	.86	44.2	18.8	23.0	23.0
4	131	.23	1.234	4.63	.82	.83	.90	44.3	20.6	23.6	23.7
5	131	.33	1.319	4.19	.90	.92	.99	46.5	24.1	24.9	25.0
6	131	.48	1.393	3.88	.95	.97	1.06	47.0	23.7	25.7	25.9
7	131	.72	1.479	3.48	1.02	1.04	1.15	47.6	24.6	26.9	27.3
8	131	1.12	1.513	3.07	1.09	1.13	1.25	47.9	30.1	28.4	29.1
9	131	2.15	1.532	2.63	1.16	1.23	1.36	46.8	24.7	29.2	30.4
10	131	23.49	1.608	2.16	1.27	1.44	1.63	34.8	28.5	32.2	34.6
All	1,308	2.87	1.306	3.88	.95	.98	1.08	45.2	23.1	25.8	26.3
Panel B: *1981 until 1998 (Recent Period)*											
1	107	.03	.581	5.86	.71	.72	.70	46.4	16.1	13.3	13.2
2	108	.10	.650	6.17	.74	.74	.73	42.7	16.0	13.3	13.1
3	108	.19	.674	6.17	.72	.72	.71	39.0	18.1	13.5	13.3
4	108	.31	.690	6.09	.76	.77	.76	39.1	17.0	13.7	13.6
5	108	.47	.744	5.77	.77	.79	.77	38.0	18.4	13.7	13.6
6	108	.74	.788	5.42	.81	.83	.81	37.7	17.3	13.9	13.8
7	108	1.21	.828	5.07	.85	.89	.87	37.4	18.0	14.1	14.0
8	108	2.34	.880	4.62	.91	.97	.95	36.5	20.4	14.5	14.5
9	108	6.71	.923	4.18	.96	1.10	1.07	33.4	23.8	14.8	14.8
10	108	45.58	.766	3.60	1.11	1.15	1.08	20.3	22.3	15.8	13.8
All	1,079	5.75	.753	5.29	.83	.87	.85	37.1	18.7	14.1	13.8

Table V

Portfolio Results for the Relation between Probability of Bankruptcy, Size, Book-to-Market, OLS Beta, Bankruptcy-Adjusted Beta, Correlation with the Market and Year-Ahead Returns: Actual and According to Models *Z* and *P*
- Nasdaq Firms -

Average characteristics are given from January 1973 to December 1980 and from January 1981 to January 1999 for each of ten portfolios and for all portfolios. • Firms are sorted in order of probability of bankruptcy at the beginning of each month and assigned to portfolios 1 through 10 so that each portfolio contains one decile. Portfolio 10 has the most distressed firms. • The main observation is that expected returns according to model *P* (that admits bankruptcy) and model Z (that ignores bankruptcy) diverge at the high end of failure probabilities during the more recent period. • *N* is the average number of firms in the portfolio; *p* is probability of bankruptcy from Ohlson's [1980] model, with the required data assumed to be available at fiscal year end; *BME* is the ratio of book to market equity, both publicly available at the time (it is assumed that information is available six months after fiscal year end); Size is log of market equity at the end of the most recent known fiscal year in millions of dollars; OLS beta (β^Z) is based on the 60 preceeding months (at least 24 required) using one Dimson lag [1979], and using Shumway [1997] and Shumway and Warther's [1999] corrections for missing delisting returns; model *P* betas (β^P) are estimated from β^Z using the procedure described in Section III; model *P conditional* betas are obtained by multiplying β^P by the ratio of conditional to unconditional expected dividends (*k*); ρ *is* the correlation between cash flows and the market, inferred from β^Z using the procedure described in Section III; actual returns are compound returns accumulated over the twelve months following portfolio formation, including delisting returns, with missing delisting returns replaced by –30% and –50% respectively for NYSE-AMEX and Nasdaq firms; expected returns under model *Z* are twelve month returns computed with the traditional CAPM formula and β^Z; expected returns under model *P* are computed with expression (11).

Portfolio	N	Firm Characteristics			Beta			ρ	Actual Returns	Model Returns	
		p (%)	BME	Size	β^Z	β^P	Conditional β^P	(%)		*Z* (% pa)	*P* (% pa)
					Panel A: *1973 until 1980 (Early Period)*						
1	56	.03	1.064	2.92	.87	.87	.93	56.4	18.1	23.2	22.8
2	56	.14	1.353	2.55	.92	.92	1.00	51.4	27.8	24.6	24.5
3	56	.30	1.468	2.43	1.00	1.01	1.09	51.5	23.8	25.9	25.9
4	56	.49	1.396	2.51	.99	1.01	1.12	48.5	25.0	26.6	26.7
5	56	.73	1.613	2.49	.97	.99	1.10	45.3	24.5	26.4	26.8
6	56	1.05	1.605	2.36	1.03	1.07	1.18	45.8	21.7	26.8	27.3
7	56	1.70	1.550	2.14	1.10	1.16	1.25	45.6	21.4	26.7	27.5
8	56	2.99	1.618	1.90	1.13	1.22	1.36	43.6	21.5	28.5	29.9
9	56	7.97	1.679	1.54	1.22	1.40	1.57	39.6	24.0	29.7	32.8
10	56	43.97	1.507	1.49	1.30	1.45	1.60	25.3	17.9	30.5	31.0
All	560	5.89	1.486	2.24	1.05	1.11	1.22	45.3	22.6	26.9	27.5
					Panel B: *1981 until 1998 (Recent Period)*						
1	116	.02	.617	4.16	.85	.85	.84	55.3	15.1	14.2	14.1
2	117	.11	.721	3.97	.94	.94	.93	52.7	17.3	15.0	14.8
3	117	.31	.776	3.66	.98	.99	.96	49.3	14.3	14.9	14.6
4	117	.68	.871	3.54	1.00	1.02	.99	46.0	15.4	15.1	14.9
5	117	1.34	.902	3.28	1.03	1.07	1.05	43.5	15.3	15.6	15.4
6	117	2.82	.908	3.00	1.07	1.14	1.10	40.6	18.7	15.2	15.1
7	117	7.04	.927	2.71	1.11	1.26	1.21	36.5	14.6	15.6	15.4
8	117	19.26	.905	2.63	1.23	1.52	1.41	32.3	12.1	16.3	15.5
9	117	49.16	.700	2.39	1.30	1.58	1.46	22.4	10.1	17.5	15.5
10	116	90.53	.437	2.55	1.42	.58	.56	5.6	3.9	18.1	11.6
All	1,166	17.11	.777	3.19	1.09	1.10	1.05	38.4	13.7	15.8	14.7

Table VI

Regression Results 1973-1998
- NYSE, AMEX and Nasdaq Firms -

Regressions are run monthly from January 1973 until January 1999. The coefficients are averages of coefficients from the monthly cross sections. The *t*-statistic (in brackets) is the average coefficient divided by its time-series standard error. • The dependent variable is actual returns (r) accumulated during the twelve months that follow each cross section, including delisting returns and adjustments for missing delisting returns; OLS beta (β^Z) is based on the 60 preceeding months (at least 24 required) using one Dimson lag [1979] and Shumway [1997] and Shumway and Warther's [1999] corrections for missing delisting returns; model *P* betas (β^P) are estimated from β^Z using the procedure described in Section III; K^* is the ratio of conditional to unconditional expected dividends; B^* is β^P multiplied by K^*; *BME* is the ratio of book to market equity, both publicly available at the time; Size is log of market equity at the end of the most recent known fiscal year in millions of dollars; *P* is probability of bankruptcy computed with Ohlson's [1980] model.

Panel A: *Traditional Models (Z)*

r =	a_0 +	$a_1\,\beta^Z$	+	$a_3\ S$ +	$a_4\ BME$ +	$a_5\ P$
Pred. signs	(+)	(+)		(-)	(+)	(-)
Model Z1	.203	-.017				
	[21.0]	[-2.7]				
Model Z2	.300	-.042		-.020		
	[19.5]	[-6.9]		[-10.7]		
Model Z3	.144	-.015			.065	
	[14.4]	[-2.4]			[16.3]	
Model Z4	.195			-.013	.054	
	[9.4]			[-6.6]	[12.1]	
Model Z5	.219	-.031		-.013	.054	
	[12.2]	[-5.6]		[-6.7]	[12.0]	
Model Z6	.236	-.027		-.016	.050	-.078
	[13.6]	[-5.0]		[-8.5]	[11.8]	[-7.2]

Panel B: *Positive Risk of Bankruptcy Models (P)*

r =	b_0 +	$b_1\,K^*$ +	$b_2\,B^*$ +	$b_3\ S$ +	$b_4\ BME$ +	$b_5\ P$
Pred. signs	(-)	(+)	(+)	(?)	(?)	(?)
Model P1	-.514	.690	.041			
	[-6.3]	[8.6]	[4.4]			
Model P2	-.506	.779	.016	-.020		
	[-6.4]	[10.1]	[1.6]	[-10.8]		
Model P3	-.442	.568	.032		.066	
	[-5.4]	[7.2]	[3.6]		[16.3]	
Model P5	-.444	.644	.006	-.014	.055	
	[-5.8]	[8.7]	[.7]	[-6.9]	[11.7]	
Model P6	-.098	.331	-.034	-.017	.050	-.088
	[-1.4]	[4.9]	[-4.6]	[-10.9]	[11.6]	[-7.1]

[1] For a sample of firms listed in the NYSE or AMEX Dichev reported that the portfolio consisting of the one tenth most distressed firms earned 0.48 % per month (0.17 % for the Nasdaq) on average over a period extending from 1981 until 1995, when the overall market's return reached 0.91 % per month. (See Dichev [1998], Table I, page 1136 and Table III, page 1139.)

[2] Allowing the return on the market to be defined on the entire real line implies no loss of realism, since the actual probability of $\tilde{r}_m$ even approaching –1 is minuscule (with $\bar{r}_m = .15$ and $s_m = .23$, $Pr\{\tilde{r}_m < -1\} = 2.87 \times 10^{-7}$.) However the resulting simplification in the algebra of covariances between $\tilde{X}$ and $\tilde{r}_m$ is considerable because only $\tilde{X}$ needs to be treated as a truncated random variable.

[3] See Haley and Schall (1979), pp. 97-99, and footnote 14 on page 97.

[4] The derivation of the formula for expected dividends uses results concerning the expected value of truncated normal distributions which are given in Maddala [1983].

[5] It does hold if $\rho = 0$, in which case the probability of bankruptcy is independent of the realized return on the market.

[6] From numerical examples introduced in Section III, we will also observe that conditional expected returns increase (decrease) at an increasing rate with the realized return on the market given a positive (negative) correlation of cash flows with the market.

[7] As we will see in Section III, ordinary least squares do not lead in general to an unbiased estimate of systematic risk. The symbol β^{P} in this context stands for an unbiased measure of systematic risk.

[8] Op. cit., Chapter 5, pages 168-173.

[9] Barra, Inc. provides portfolio and enterprise risk management systems to investment professionals. The Barra Aegis System™ is dedicated to equities and the Barra Cosmos System™ to fixed income securities.

[10] Op. cit., Chapter 8, pages 272-273.

[11] Foster [1986] contains a description of these techniques.

[12] A well known example is Fama and French's [1992] influential paper demonstrating the failure of OLS portfolio betas to capture cross-sectional variations in common stock returns.

[13] It would have been more precise to condition on at most one occurrence of $\tilde{r}_e = -1$. I have chosen the strict inequality formulation because it simplifies the algebra but does not compromise realism.

[14] Model *P* is associated with expression (8) for systematic risk, (10) for unconditional and (11) for conditional expected returns. Model *Z* is associated with expression (14) for systematic risk and (10) – with β^Z substituted for β^P – for both conditional and unconditional expected returns. There is no value of *p* for which models *P* and *Z* coincide, which means that *Z* is not a special case of *P*. We can however describe *Z* as an approximation that holds over certain ranges of the parameters (e,g., when the probability of bankruptcy is "small").

[15] A related problem is discussed by G. S. Maddala [1983] under the title "problems of aggregation", in Chapter 6, pages 182-185.

[16] The bisection method is a good option because it is easy to implement and is guaranteed to produce a solution if a solution can be bracketed at the beginning of the process. A useful reference on this and other algorithms is Press, Flannery, Teukolsky, and Vetterling's [1986] Numerical Recipes. The bisection method is described in Chapter 9, pages 243-247.

[17] Cambell, Lo and MacKinlay [1997] discuss these issues on pages 13-20.

[18] Since traditional tests of whether the lognormal or the truncated normal better represents stock returns suffer from the same lack of data that afflicts estimates of beta, we must leave the appropriateness of this assumption to be decided by the quality of its predictions regarding systematic risk and returns.

[19] A good reference on the tobit estimator is G. S. Maddala [1983], Chapter 6, pages 182-185.

[20] This is also the model used by Dichev [1998]. The variable here denoted *JO* corresponds to Dichev's *O*.

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[21] Unlike Dichev [1998], where tests are based on probability rankings and no adjustment needs to be made for the price-level index, I require actual probabilities to calculate bankruptcy-adjusted betas and expected returns under model *P*. Therefore in this paper the *lnTA* variable is calculated as described in Ohlson [1981], including the GNP price-level adjustment.

[22] When data on funds provided by operations (FU) is not available, but cash provided by operations is, I estimate the former as follows: $FU = CFO + DWK$, (CFO is COMPUSTAT's item # 308); DWK is the year on year change in WK; and $WK = CA - CL + STDEBT - CASH$ (CA is current assets, CL is current liabilities, $STDEBT$ is item # 44, and $CASH$ is item # 1.)

[23] Dichev [1998], first paragraph on page 1135.

[24] Dichev [1998], page 1141.

[25] Tests performed with trimmed returns are in agreement with Dichev [1980] in producing below average returns for the NYSE-AMEX in the more recent period.

[26] Dichev [1980], page 1142.

[27] See Greene [2000], Chapter 3, page 83

[28] Maddala[1983], Appendix, page 365.